Last time: Additive Haar measure $\mu(\mathbb{Z}_p)=1.$

ie on $(\mathbb{Q}_p,+)$, $\mu(\underset{\mathbb{Q}_p}{a}+p^n\mathbb{Z}_p)=p^{-n}$, $n\in\mathbb{Z}$ (dx)

$= \int_{a+p^{n}\mathbb{Z}_p} 1\cdot dx$

Multiplicative Haar measure $(d\mu^\times)$

$$\int_{\mathbb{Q}_p^\times} f(x)\, d^\times x = \int_{\mathbb{Q}_p^\times} f(x) \frac{dx}{|x|_p}\cdot\frac{p}{p-1} \qquad \left(\mu^\times(\mathbb{Z}_p^\times)=1\right).$$

Additive char: $\underset{\mathbb{Q}_p}{x} \mapsto e^{2\pi i x}$

$$x = a_{-N}p^{-N}+\dots+a_{-1}p^{-1}+\underbrace{a_0+a_1p+a_2p^2+\dots}_{\mathbb{Z}_p} \mapsto e^{2\pi i(a_{-N}p^{-N}+\dots+a_{-1}p^{-1})},$$

$a_j\in\{0,\dots,p-1\}.$

Magic Fourier transform

$\mathcal{S}$ (analogue of Schwartz) loc const, cpt supp. $\quad$ supp f = cpt. $\Updownarrow$

Ex: $\mathbb{Z}_p \setminus \{0\} = \bigsqcup_{n \geq 0} p^n \mathbb{Z}_p^\times$, $f = |\cdot|_p$ on $\mathbb{Z}_p \setminus \{0\}$ is loc constant.

$\underline{\text{Supp}} \subset \mathbb{Z}_p$, $\overline{\text{Supp}} = \mathbb{Z}_p$. But f is not a finite sum of indicator functions.

By $\mathbb{R}$-analogue:

K cpt

Then $f = \sum_{j=1}^{N} c_j \mathbb{1}_{\{a_j + p^{n_j} \mathbb{Z}_p\}}$,

f loc const, cptly supported, $f : \mathbb{Q}_p \to \mathbb{C}$

Def: $\hat{f}(\xi) := \int_{\mathbb{Q}_p} f(x) e^{2\pi i x \cdot \xi} \, dx$.

$\xi \in \mathbb{Q}_p$.

Ex: $\widehat{\mathbb{1}_{\{a + p^n \mathbb{Z}_p\}}}(\xi) = \int_{x \in \mathbb{Q}_p} \mathbb{1}_{\{a + p^n \mathbb{Z}_p\}} e^{2\pi i x \cdot \xi} \, dx$

$$= \int_{x \in a + p^n \mathbb{Z}_p} e^{2\pi i x \cdot \xi} \, dx \overset{\text{Haar}}{=} \underline{\underline{\int_{x \in p^n \mathbb{Z}_p} e^{2\pi i (x+a)\cdot \xi} \, dx}}$$

$$\overset{\text{char}}{=} e^{2\pi i a \cdot \xi} \underbrace{\int_{x \in p^n \mathbb{Z}_p} e^{2\pi i x \cdot \xi} \, dx}_{} = \begin{cases} 0 & \xi \notin p^{-n} \mathbb{Z}_p \\ \dfrac{e^{2\pi i a \cdot \xi}}{p^n} & \text{else}, \end{cases}$$

$x = a_n p^n + a_{n+1} p^{n+1} + \cdots$, a_j free.

<u>Case 1:</u> If $\underbrace{|x \cdot \xi|_p \le 1}_{} \ \underline{\underline{(\forall x)}}$, i.e. $x \cdot \xi_p \in \mathbb{Z}_p \ \forall x \in p^n \mathbb{Z}_p$,

$|x|_p \cdot |\xi|_p =$

then $e^{2\pi i x \cdot \xi} = 1$. $\Rightarrow \underbrace{\int e^{\cdots} dx}_{} = \boxed{p^{-n}}$.

$|x|_p \le p^{-n}$, $\Leftrightarrow |\xi|_p \le p^n$, $\Leftrightarrow \underbrace{\xi \in p^{-n} \mathbb{Z}_p}_{}$.

<u>Case 2:</u> $\xi \notin p^{-n} \mathbb{Z}_p$, then $\exists x \in p^n \mathbb{Z}_p$ s.t.

$|x \cdot \xi|_p > 1 \Rightarrow e^{2\pi i x \cdot \xi}$ varies with x.

Does so over complete set of residues mod p.

$\Rightarrow S = 0.$

Dmi: $\widehat{\mathbb{1}_{\{a + p^n \mathbb{Z}_p\}}}(\xi) = e^{2\pi i a \cdot \xi} \cdot p^{-n} \cdot \mathbb{1}_{p^{-n}\mathbb{Z}_p}(\xi)$

Calculate: $\int_{\xi \in \mathbb{Q}_p} \hat{f}(\xi) e^{-2\pi i x \cdot \xi} d\xi.$

(When $f = \mathbb{1}_{\{a + p^n \mathbb{Z}_p\}}$)

$$= \int_{\xi \in \mathbb{Q}_p} e^{2\pi i a \cdot \xi} p^{-n} \mathbb{1}_{\xi \in p^{-n}\mathbb{Z}_p} e^{-2\pi i x \cdot \xi} d\xi.$$

$$= p^{-n} \int e^{2\pi i (a - x) \cdot \xi} d\xi$$

$\xi \in p^{-n}\mathbb{Z}_p$

$$= \begin{cases} 1, & a-x \in p^n\mathbb{Z}_p \\ 0, & \text{else} \end{cases}$$

$$= \mathbb{1}_{a+p^n\mathbb{Z}_p}(x) = f(x).$$

The "$\underline{a}$deles" ("$\underline{a}$dditive ideles")

$$\mathbb{A} \overset{?}{=} \mathbb{R} \times \prod_p \mathbb{Q}_p, \quad \text{Cartesian product.}$$

$$x = ((x_\infty : \mathbb{R}), (x_2 : \mathbb{Q}_2), (x_3 : \mathbb{Q}_3), (x_5 : \mathbb{Q}_5), \ldots),$$

$$|x|_{\mathbb{A}} = |x_\infty|_\infty \cdot \prod_p |x_p|_p$$

$q \mapsto (q : \mathbb{R})$

$\mathbb{Q} \subset \mathbb{A}, \; q \mapsto (q, q, q, \ldots)$

Recall: $\forall q \in \mathbb{Q}\setminus\{0\}$, $|q|_{\mathbb{A}} = \prod_{p \le \infty} |q|_p = 1$.

$$\left((1 : \mathbb{R}), \left(\tfrac{1}{2} : \mathbb{Q}_2\right), \left(\tfrac{1}{3} : \mathbb{Q}_3\right), \left(\tfrac{1}{5} : \mathbb{Q}_5\right), \ldots \right).$$

$$|\cdot|_{\mathbb{A}} = \infty,$$

Take "random" $x \in [-1,1] \times \prod_p \mathbb{Z}_p$.

$$\Rightarrow \quad |x|_{\mathbb{A}} = 0.$$

Rmk: Cartesian product not locally cpt, no nice Haar measure, Fourier theory ...

Instead! $\mathbb{A} = \mathbb{R} \times \prod_p \mathbb{Q}_p$

"restricted" product i.e.

$$\mathbb{A} = \left\{ x = (x_\infty, x_2, x_3, \ldots) \;\middle|\; \begin{array}{l} x_p \in \mathbb{Q}_p, \ p \le \infty \\ \text{all but finitely many} \\ x_p \in \mathbb{Z}_p \end{array} \right\},$$

Does $\mathbb{Q}$ still embed in here? Yes, finitely many things are in the denominator.

This $\mathbb{A}$ is loc cpt, under topology basis open sets: $|S| < \infty$.

$$\prod_{p \in S} U_p \times \prod_{p \notin S} \mathbb{Z}_p.$$

Can I add adeles?

$x+y = (x_\infty + y_\infty,\ x_2+y_2,\ x_3+y_3, \ldots)$

eventually $x_p + y_p \in \mathbb{Z}_p$ which is a ring.

Can I multiply?

$x \cdot y = (x_\infty \cdot y_\infty, \ldots)$

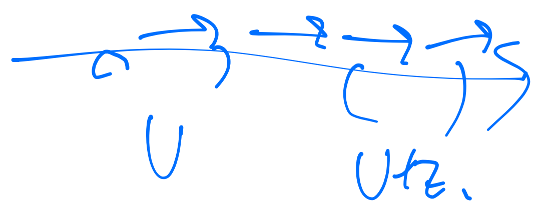
~~$x \div y = (x_\infty / y_\infty, \ldots)$~~

$\Downarrow$

0

"Ring of adeles", $\mathbb{A} \supset \mathbb{Q}$

Recall: $\mathbb{Z} \circlearrowright \mathbb{R}$



Given $q \in \mathbb{Q}$: $x \mapsto x+q$. ($\mathbb{A}$)

Action of $\mathbb{Z}$ on $\mathbb{R}$ is discrete, i.e.

around every pt $z \in \mathbb{Z}$, $\exists$ U open $\ni z$, and no other pts of $\mathbb{Z}$. Eg:

$z=0$, $U = (-1,1)$.

What about $\mathbb{Q} \subset \mathbb{A}$, is it ~~dense~~? $\boxed{\text{discrete?}}$

Can I find an open set $U \subseteq \mathbb{A}$ around $0 \in \mathbb{Q}$

containing <u>no</u> other rationals.

What about $U := (-1,1) \times \prod_p \mathbb{Z}_p$. open.

$$|\cdot|_{\mathbb{A}}\Big|_U < 1. \quad \text{But } \forall q \in \mathbb{Q}\setminus\{0\},\ |q|_{\mathbb{A}} = 1.$$

So U contains <u>no</u> other rationals.

For $\mathbb{Z} \subset \mathbb{R}$, Find domain for this

action, i.e. $D \subset \mathbb{R}$, is open, $\mathbb{R}/\mathbb{Z} = \bigcirc$

$\forall r \in \mathbb{R}$, $\exists z \in \mathbb{Z}$ s.t. $z + r \in \overline{D}$. $(0,1) = [0,1)$

& if $r, r' \in D$ & $r = r' + z$ for some $z \in \mathbb{Z}$,
$\Rightarrow r = r'$ ($\&\ z = 0$).

$\Gamma \circlearrowright X$ discrete group action, Fund dom
group

$\mathbb{Z} \to \Gamma$, $\mathbb{R} \to X$

<u>Want:</u> fund domain for $\mathbb{Q}^{\mathbb{A}}$.

Need to be reminded of <u>Chinese Remainder</u>

<u>Thm:</u> Given $p_1, \dots, p_k$ distinct primes, any $e_1, \dots, e_k \in \mathbb{N}$, any $c_1, \dots, c_k$, $c_j \in \mathbb{Z}/p_j^{e_j}\mathbb{Z}$,

$\exists x \in \mathbb{Z}$ s.t.
$$\left.\begin{aligned} x &\equiv c_1 \bmod p_1^{e_1} \\ x &\equiv c_2 \bmod p_2^{e_2} \\ &\vdots \\ x &\equiv c_k \bmod p_k^{e_k} \end{aligned}\right\}$$

Eg: $\underset{c_1}{x \equiv 2} \bmod \underset{p_1}{3}^{2 = e_1}$, $2 \bmod 5$, $5 \bmod 7$

Let $N = 3^2 \cdot 5 \cdot 7$, let $N_j = \dfrac{N}{p_j^{e_j}}$

$= \prod_{j=1}^{a} p_j^{e_j}$ Then $(N_j, p_j^{e_j}) = 1 \Rightarrow \exists\, u_j$ s.t. $N_j u_j \equiv 1 \pmod{p_j^{e_j}}$

$N_1 = 5 \cdot 7$, $N_2 = 3^2 \cdot 7$, $N_3 = 3^2 \cdot 5$.

$u_1 = -1$, $u_2 = 2$, $u_3 = -2$.

Let $x = c_1 \cdot N_1 \cdot u_1 + c_2 \cdot N_2 \cdot u_2 + \dots + c_k N_k u_k$

$= 2 \cdot \underbrace{5 \cdot 7 (-1)} + \underbrace{2 \cdot 3^2 \cdot 7 \cdot 2} + \underbrace{5 \cdot 3^2 \cdot 5 \cdot (-2)}$,

$x \pmod 9$: $2 \cdot 1$, 0

$x \pmod 5$: 0, $2 \cdot 1$, 0

$x \pmod 7$: 0, 0, 5.