Recall:

$$\underline{Z(\mathbb{A})} GL_2(\mathbb{Q}) \backslash GL_2(\mathbb{A}) / O_2(\mathbb{R}) \times \prod_p GL_2(\mathbb{Z}_p) = SL_2(\mathbb{Z}) \backslash \mathbb{H}$$

" $\left\{ \begin{pmatrix} a & 0 \\ 0 & a \end{pmatrix} : a \in \mathbb{A}^\times \right\}$

"aut rep is an irred component of

$$L^2\left(Z(\mathbb{A}) GL_2(\mathbb{Q}) \backslash GL_2(\mathbb{A}) \right)$$

w.r.t. $GL_2(\mathbb{A}_f)$ + invariant diff operators at ∞."

$$\underbrace{\hspace{6cm}}_{\text{"}(\mathfrak{g}, K)\text{-modules"}}$$

$G = SL_2(\mathbb{R}) = NAK$, $n_x = \begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix}$, $a_y = \begin{pmatrix} \sqrt{y} & 0 \\ 0 & \frac{1}{\sqrt{y}} \end{pmatrix}$ $\begin{pmatrix} y & 0 \\ 0 & 1 \end{pmatrix}$

$f(\underline{n_x a_y k_\theta})$ $k_\theta = \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix}$

$f : G \to \mathbb{C}$.

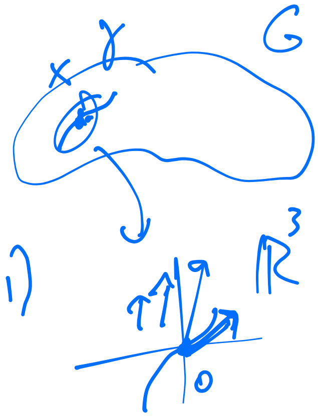
Linear differential operators: $(2x^2 + \theta) \partial_x f + y \partial_\theta f (x, \theta)$

Care about $D \in \mathcal{D} = \{ \underline{\text{invariant}}$ differential operators $\}$

$$\pi(g)(Df) = D(\pi(g).f)$$

$SL_2(\mathbb{R})$: G = Lie group = $\overset{\text{smooth}}{(?)\ \text{manifold}}$

w/ group structure,



$\gamma : (-1,1) \to G$

$\gamma'(0)$

Key: $\underbrace{T_e G}_{\mathfrak{g}} = \left\{ \begin{array}{l} C^1\text{-curves} \\ \gamma : (-1,1) \to G \\ \gamma(0) = e \end{array} \right\} \Big/ \sim$

$\gamma_1 \sim \gamma_2 \Leftrightarrow \gamma_1'(0) = \gamma_2'(0)$.

If $\gamma(t) = \begin{pmatrix} 1 + at + o(t^2) & 0 + bt + o(t^2) \\ 0 + ct + o(t^2) & 1 + dt + o(t^2) \end{pmatrix} \in G$.

i.e. $1 = \det \gamma \simeq (1+at)(1+dt) - \cancel{bct^2} = 1 + \underline{\underline{(a+d)}}t + \ldots$

$\gamma(t) \simeq I + X$, $X = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$, need $\operatorname{tr} X = 0$.

$\mathfrak{g} = \mathfrak{sl}_2(\mathbb{R}) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in M_2(\mathbb{R}) : a + d = 0 \right\}$.

Lie vector space.

$\mathfrak{g} \xrightarrow{d} D : X \mapsto dX := \frac{d}{dt} f(g \exp(tX))\Big|_{t=0}.$

$\exp : \mathfrak{g} \to G : X \mapsto I + X + \frac{X^2}{2} + \frac{X^3}{3!} + \frac{X^4}{4!} + \dots$

$\exp(A+B) \overset{?}{=} \exp(A) \cdot \exp(B)$. Not nec!

Lemma: $\det(\exp X) = \exp(\overset{0}{\operatorname{tr}} X) = 1.$

$\hookrightarrow e^{\lambda_1} \cdot e^{\lambda_2}$ $\qquad \hookrightarrow e^{\lambda_1 + \lambda_2}.$

Hint: $X = \begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix} \Rightarrow \exp X = \begin{pmatrix} e^{\lambda_1} & \\ & e^{\lambda_2} \end{pmatrix}.$

$\frac{d}{dt} f(g \underbrace{(I + tX)}_{\not\in G.})\Big|_{t=0}.$ (ok for $GL_2(\mathbb{R})$), $f : G \to \mathbb{C}$.

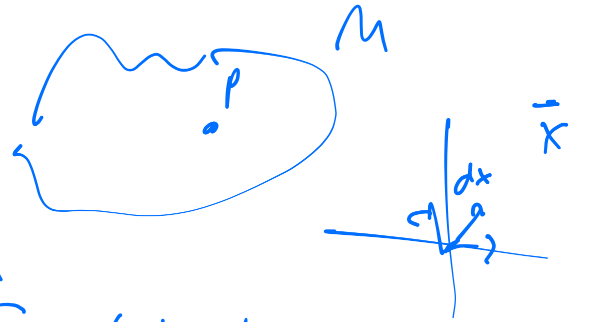
Lemma: $\mathfrak{g} \to D : X \to dX$ is linear.

Check: $d(aX) = \frac{d}{dt} f(g \exp(t \cdot \underbrace{a \cdot X}_{\in \mathfrak{g}}))\Big|_{t=0} = a\,dX.$

Check $d(X+Y) = \frac{d}{dt}\left(f(g \exp(t(X+Y)))\right)\Big|_{t=0} \overset{?}{=} dX + dY.$

In general manifold, local chart $\vec{x} = (x_1, \dots, x_n)$.

T_pM $\quad dx_1, \dots, dx_n$.



$$D = \sum_{j=1}^{n} f_j(\vec{x}) \cdot dx_j, \quad E = \sum_{k=1}^{n} g_k(\vec{x}) \cdot dx_k.$$

$$D(E) = \sum_{j=1}^{n}\sum_{k=1}^{n} f_j(\vec{x}) \frac{\partial}{\partial x_j}\left[g_k(\vec{x}) \cdot dx_k \right].$$

$$= \sum_{j=1}^{n}\sum_{k=1}^{n} \underline{f_j(\vec{x})} \left[\underbrace{\frac{\partial}{\partial x_k} g_k(\vec{x}) \cdot dx_k}_{\text{linear}} + \underbrace{\underline{g_k(\vec{x})}\, dx_{jk}}_{\text{2nd order}} \right]$$

$$-E(D) = \sum_{j=1}^{n}\sum_{k=1}^{n} \underline{g_k(\vec{x})} \left[\frac{\partial}{\partial x_j} f_j(\vec{x}) \cdot dx_j + \underline{f_j(\vec{x})}\, dx_{jk} \right].$$

$DE - ED = \underline{\underline{\text{linear}}}$.

$\mathfrak{g} = T_eG$ vector space, $\rightarrow$ Lie algebra.

$$[X,Y] = \underbrace{X\cdot Y - Y\cdot X}_{\sim} \in \mathfrak{g}.$$

pf: $$\operatorname{tr}\left(\begin{pmatrix}a&b\\c&-a\end{pmatrix}\begin{pmatrix}d&e\\f&-d\end{pmatrix} - \begin{pmatrix}d&e\\f&-d\end{pmatrix}\begin{pmatrix}a&b\\c&-a\end{pmatrix}\right)$$

$$= \cancel{ad} + \cancel{bf} + \cancel{ce} + \cancel{ad} - (\cancel{da} + \cancel{ec} + \cancel{bf} + \cancel{ad}) = 0.$$

Commutative? No $\quad [Y,X] = -[X,Y].$

associative? No $\quad [[X,Y],Z] - [X,[Y,Z]] = [Y,[Z,X]].$

Jacobi: $\quad [[X,Y],Z] + [[Y,Z],X] + [[Z,X],Y] = 0.$

exercise: check this when $[X,Y] = \underline{XY - YX}$.

Examples: $\mathfrak{g} \ni \begin{pmatrix}0&1\\0&0\end{pmatrix} = R$, nilpotent

$R^2 = 0.$

$$\exp(xR) = I + xR + \cancel{\frac{x^2R^2}{2}} = \underbrace{\begin{pmatrix}1&x\\0&1\end{pmatrix}}_{\text{unipotent}} = n_x = u_x$$

$$H = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} = a. \qquad \exp(tH) = \begin{pmatrix} e^{t} & 0 \\ 0 & e^{-t} \end{pmatrix} = a_t.$$

$$L = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} \qquad \exp(xL) = \begin{pmatrix} 1 & 0 \\ x & 1 \end{pmatrix}.$$

$$Y = \begin{pmatrix} 0 & +1 \\ -1 & 0 \end{pmatrix}, \quad Y^2 = \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}, \quad Y^3 = \begin{pmatrix} 0 & -1 \\ +1 & 0 \end{pmatrix}, \quad Y^4 = 1,$$

$$\exp(\theta Y) = I + \theta Y + \frac{\theta^2(-I)}{2} - \frac{\theta^3}{3!} Y + \cdots$$

$$= \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix} = k_\theta. \qquad Y \in \mathfrak{g}.$$

$S \in \Gamma < G.$

$U(\mathfrak{g})$ = "universal enveloping algebra" of $\mathfrak{g}$.

$$\otimes \mathfrak{g} = \bigoplus_{k=0}^{\infty} \otimes^k \mathfrak{g} \qquad \otimes^k \mathfrak{g} = \mathfrak{g} \otimes_{\mathbb{R}} \cdots \otimes \mathfrak{g}.$$

$$X \in \otimes^k \mathfrak{g}, \quad Y \in \otimes^l \mathfrak{g}, \quad X \otimes Y \in \otimes^{k+l} \mathfrak{g}.$$

I ideal gen by $X \otimes Y - Y \otimes X - [X, Y]$ for $X, Y \in \mathfrak{g}$.

$$\mathcal{U}(\mathfrak{g}) = \bigotimes \mathfrak{g} / I \ \ni \ D = D_{X_\alpha} \cdots D_{X_3} D_{X_2} D_{X_1}$$

$Z(\mathcal{U}(\mathfrak{g}))$ = center of enveloping algebra

$$= \{ D \in \mathcal{U}(\mathfrak{g}) : DE = ED \ \ \forall E \in \mathcal{U}(\mathfrak{g}) \}$$

$Z(\mathcal{U}(\mathfrak{g}))$ gen by $\mathcal{C}$ = Casimir operator

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$$[H, R] = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} - \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

$$= \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} - \begin{pmatrix} 0 & -1 \\ 0 & 0 \end{pmatrix} = 2R.$$

$$[H, L] = -2L,$$

$$[R, L] = \overbrace{\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}\begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}}^{\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}} - \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$$

$$= H$$

$[\,,\,]$	H	R	L
H	0	$2R$	$-2L$
R	$-2R$	0	H
L	$2L$	$-H.$	0

$-4\,C := H^2 + 2\underline{RL} + 2LR \in \mathcal{U}(\mathfrak{g})$, $\in \otimes^2 \mathfrak{g}$.

<u>not</u> as matrices.

<u>claim</u>: $C \in Z(\mathcal{U}(\mathfrak{g}))$. (in fact, generates).

<u>pf</u>: check C commutes with R, L, H.

<u>Thm</u>: If $D \in Z(\mathcal{U}(\mathfrak{g}))$, then D commutes with right regular rep. G.

i.e. $\big(\pi(g)(Df)\big)(h) = (Df)(hg)$.

$= D\big(f(hg)\big)$. i.e. $\pi(g).D = D.\pi(g)$.

<u>Rmk</u>: left-reg rep commutes with $\mathfrak{g}$.

$$\frac{d}{dt} f(g \cdot h \cdot \underline{\exp(tX)})\Big|_{t=0} = \lambda_g \cdot dX = dX \cdot \lambda_g.$$

<u>Lemma</u>: let $\phi: G \times \mathbb{R} \to \mathbb{C}$ & suppose

$\exists X \in \mathfrak{g}$ s.t. ② $\frac{d}{dt}(\phi(g,t)) = dX(\phi(g,t))$.

& ① $\phi(g,0) = 0$ Then $\phi(g,t) \equiv 0$.

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pf Thm from Lemma: need: $\forall g \in G$, $D \circ \pi(g) = \pi(g) \circ D$.

in neighborhood of e, $g \rightsquigarrow \exp tX$.

Let $\phi(h,t) := (D\,\pi(\exp tX) f - \pi(\exp tX) D f)(h)$

① ✓ ② ? $\frac{d}{dt}\phi(h,t)\Big|_{t=0} = D \circ dX - dX \circ D = 0$.

Exercise: If $f(n_x a_y k_\theta)$,

$$-4\,\mathcal{C} f = \left(-y^2(\partial_{xx} + \partial_{yy}) + y \cdot \partial_x \partial_\theta\right) f.$$

If $f(g k_\theta) = f(g) \Rightarrow \underline{\underline{\mathcal{C}}} f = \underline{\underline{\Delta}} f$.

f on G/K.

f on G

If $f(gk_\theta) = e^{ki\theta} f(g)$. $\quad$ $f = \bigoplus f_k \leftarrow$ K-isotypic component.

then $\mathcal{C} f = -y^2(\partial_{xx} + \partial_{yy}) + y\partial_x (ik) = \Delta_k$.

If $f(gk_\theta) = e^{ki\theta} f(g)$,

then $(dR_+ f)(gk_\theta) = e^{(2+k)i\theta} f(g)$.