Recall: if φ on $\Gamma_0(N)$, $\varphi = \sum a_n W_{\lambda,n}(y)\, e(nx)$.

$\forall (D,N)=1$, χ prim mod D.

$$\varphi_\chi(z) = \sum a_n \chi(n) W_{\lambda,n}(y)\, e(nx) \quad \text{on } \Gamma_1(ND^2)$$

$$\chi(n) = \frac{\chi(-1)\tau(\chi)}{D} \sum_{m(D)}{}' \overline{\chi}(m)\, e^{\frac{2\pi i m n}{D}}$$

$$\boxed{\varphi(z) = \varphi(\omega_N z)}$$

$$\boxed{\varphi_\chi(z) = \frac{\chi(-1)\tau(\chi)}{D} \sum_{m(D)}{}' \overline{\chi}(m)\, \varphi\left(\begin{pmatrix} 1 & m/D \\ 0 & 1 \end{pmatrix} z\right)},$$

$$\varphi_\chi(\omega_{ND^2} z) = \frac{\chi(-1)\tau(\chi)}{D} \sum_{m(D)}{}' \overline{\chi}(m)\, \varphi\left(\underbrace{\omega_N^{-1} \begin{pmatrix} 1 & m/D \\ 0 & 1 \end{pmatrix} \omega_{ND^2} \begin{pmatrix} 1 & -\frac{\ell}{D} \\ 0 & 1 \end{pmatrix}}\begin{pmatrix} 1 & \frac{\ell}{D} \\ 0 & 1 \end{pmatrix} z\right)$$

where $Ds + \ell m N = 1$, $\ell \equiv -\overline{m}\overline{N} \,(D)$ $\sim \underbrace{\begin{pmatrix} D & -\ell \\ -mN & s \end{pmatrix}}_{\in \Gamma_0(N)}$

Last time: φ is on $\Gamma_0(N)$

① $$\varphi_\chi(\omega_{ND^2} z) = \frac{\cancel{\chi(-1)}\tau(\chi)}{D} \underbrace{\sum_{\ell(D)}{}' \cancel{\chi(-1)}\chi(\ell)\chi(N)\, \varphi\left(\underbrace{\begin{pmatrix} D & -\ell \\ -mN & s \end{pmatrix}}\begin{pmatrix} 1 & \frac{\ell}{D} \\ 0 & 1 \end{pmatrix} z\right)}_{= \varphi_{\overline{\chi}}}$$

$$= \frac{\chi(N)\tau(\chi)^2}{D} \cdot \psi_{\bar\chi}(z).$$

$$\Rightarrow (ND^2)^{s/2} \underbrace{\Lambda(\psi \times \chi, s)}_{\int_0^\infty \psi(iy)\, y^{-1/2}\, y^s \frac{dy}{y}} \overset{\circledast}{=} (ND^2)^{\frac{1-s}{2}} \chi(N) \frac{\tau(\chi)^2}{D} \Lambda(\psi \times \bar\chi, 1-s).$$

$$y \mapsto \frac{1}{yND^2}$$

Weil: How much of this is reversible?

Thm: Assume a_n, b_n, $\exists\ \lambda = \frac{1}{4}+r^2$, N s.t. $\forall\ (D,N)=1$, $\forall\ \chi \bmod D$, primitive, $\Lambda_1(\chi, s) = (2\pi)^{-s}\, \Gamma\left(\frac{s+ir}{2}\right)\Gamma\left(\frac{s-ir}{2}\right) \sum \frac{a_n \chi(n)}{n^s}$ has analy cont, bounded in vertical strips, & FE $\circledast$

Then $\psi(z) := \sum_{n \geq 1} a_n U_{\lambda,n}(y)\, e(nx)$ on $\Gamma_0(N)$.

Pf: By inverse Mellin transforms, we get

$$\psi_\chi\left(\frac{i}{y}\right) = \frac{\chi(N)\tau^2(\chi)}{D} \psi_{\bar\chi}\left(\frac{i}{yND^2}\right).$$

$$\Rightarrow \quad \varphi_\chi(\omega_{ND^2} z) = \chi(N) \frac{\tau(\chi)^2}{D} \underbrace{\varphi_{\bar\chi}(z)}. \quad (\text{From FE})$$

expand $\varphi_{\bar\chi}$. $\rightarrow$

$$= \frac{\chi(N)\tau(\chi)}{\cancel{\tau(\bar\chi)}} \; \frac{\chi(-1)\cancel{\tau(\bar\chi)}}{D} \sum_{m(D)} \bar\chi(m)\, \psi\left(\begin{pmatrix} 1 & m/D \\ 0 & 1 \end{pmatrix} z\right)$$

expand $\varphi_\chi(\omega_{ND^2}) =$

$$\frac{\chi(-1)\tau(\chi)}{D} \sum_{\ell(D)}' \underbrace{\chi(-1)\chi(\ell)\chi(N)}\, \psi\left(\begin{pmatrix} D & -\ell \\ -mN & s \end{pmatrix}\begin{pmatrix} 1 & \frac{\ell}{D} \\ 0 & 1 \end{pmatrix} z\right)$$

$$\Rightarrow \sum_{\ell(D)}' \chi(\ell)\, \underbrace{\psi\left(\begin{pmatrix} 1 & \ell/D \\ 0 & 1 \end{pmatrix} z\right)} = \sum_{\ell(D)}' \chi_\ell\, \underbrace{\psi\left(\begin{pmatrix} D & -\ell \\ -mN & s \end{pmatrix}\begin{pmatrix} 1 & \frac{\ell}{D} \\ 0 & 1 \end{pmatrix} z\right)}$$

$m = m(\ell)$; $Ds + \ell mN = 1$.
$s = s(\ell)$

$L^2((\mathbb{Z}/D\mathbb{Z})^\times)$, all $\chi \bmod D$ form an orthonormal basis for $\uparrow$.

<u>Assume:</u> D odd prime. Then $\text{Span}_\wedge$ of <u>primitive</u> $\chi(\bmod D)$

$$L^2_0((\mathbb{Z}/D\mathbb{Z})^\times) = \chi_0^\perp. \quad \Rightarrow \forall\, c : (\mathbb{Z}/D\mathbb{Z})^\times \to \mathbb{C}$$

with $\sum'_{\ell (D)} c(\ell) = 0$, we have:

$$\sum'_{\ell(D)} c(\ell)\,\psi\left(\begin{pmatrix}1 & \ell/D\\ 0 & 1\end{pmatrix}z\right) = \sum'_{\ell(D)} c(\ell)\,\psi\left(\begin{pmatrix}D & -\ell\\ -mN & s\end{pmatrix}\begin{pmatrix}1 & \ell/D\\ 0 & 1\end{pmatrix}z\right)$$

Fix $r \in (\mathbb{Z}/D\mathbb{Z})^\times$, set $c(\ell) = \begin{cases} 1 & \ell = r\ (D) \\ -1 & \ell = -r\ (D) \\ 0 & \text{else.}\end{cases}$

$$\psi\left(\begin{pmatrix}1 & r/D\\ 0 & 1\end{pmatrix}z\right) - \psi\left(\begin{pmatrix}1 & -r/D\\ 0 & 1\end{pmatrix}z\right) = \psi\left(\begin{pmatrix}D & -r\\ -mN & s\end{pmatrix}\begin{pmatrix}1 & r/D\\ 0 & 1\end{pmatrix}z\right)$$

$$- \psi\left(\begin{pmatrix}D & r\\ mN & s\end{pmatrix}\begin{pmatrix}1 & -r/D\\ 0 & 1\end{pmatrix}z\right)$$

$Ds + mNr = 1$

$Ds + (-m)N(-r) = 1$

$$\Rightarrow \psi\left(\begin{pmatrix}D & -r\\ -mN & s\end{pmatrix}\begin{pmatrix}1 & r/D\\ 0 & 1\end{pmatrix}z\right) - \psi\left(\begin{pmatrix}1 & r/D\\ 0 & 1\end{pmatrix}z\right) \qquad \boxed{z \mapsto \begin{pmatrix}1 & r/D\\ 0 & 1\end{pmatrix}z.}$$

$$= \psi\left(\begin{pmatrix}D & r\\ mN & s\end{pmatrix}\begin{pmatrix}1 & -r/D\\ 0 & 1\end{pmatrix}z\right) - \psi\left(\begin{pmatrix}1 & -r/D\\ 0 & 1\end{pmatrix}z\right).$$

$$\Rightarrow \psi\left(\begin{pmatrix} D & -r \\ -mN & s \end{pmatrix}\begin{pmatrix} 1 & \frac{2r}{D} \\ 0 & 1 \end{pmatrix} z\right) - \psi\left(\begin{pmatrix} 1 & 2r/D \\ 0 & 1 \end{pmatrix} z\right)$$

$$= \psi\left(\begin{pmatrix} D & r \\ mN & s \end{pmatrix} z\right) - \psi(z). \qquad \boxed{\begin{pmatrix} D & -r \\ -mN & s \end{pmatrix}^{-1} = \begin{pmatrix} s & r \\ mN & D \end{pmatrix}}$$

This holds as long as $(D, 2N) = 1$, $\underline{D = \text{prime}}$

$r \in (\mathbb{Z}/D)^\times$. Similarly, $\forall$ $(s, 2N) = 1$, $\underline{s = \text{prime}}$,

$$\psi\left(\begin{pmatrix} s & -r \\ -mN & D \end{pmatrix}\begin{pmatrix} 1 & \frac{2r}{s} \\ 0 & 1 \end{pmatrix} z\right) - \psi\left(\begin{pmatrix} 1 & 2r/s \\ 0 & 1 \end{pmatrix} z\right)$$

$$= \psi\left(\begin{pmatrix} s & r \\ mN & D \end{pmatrix} z\right) - \psi(z).$$

True for all z, send $z \mapsto \underline{\begin{pmatrix} 1 & \frac{-2r}{s} \\ 0 & 1 \end{pmatrix}\begin{pmatrix} D & r \\ mN & s \end{pmatrix}\cancel{\begin{pmatrix} 1 & \frac{2r}{D} \\ 0 & 1 \end{pmatrix}} z}$

$$-\psi\left(\cancel{\begin{pmatrix} 1 & \frac{2r}{D} \\ 0 & 1 \end{pmatrix}} z\right) + \psi\left(\begin{pmatrix} D & r \\ mN & s \end{pmatrix}\cancel{\begin{pmatrix} 1 & \frac{2r}{D} \\ 0 & 1 \end{pmatrix}} z\right).$$

$$= -\psi\left(\begin{pmatrix} s & r \\ mN & D \end{pmatrix}\begin{pmatrix} 1 & -\frac{2r}{s} \\ 0 & 1 \end{pmatrix}\begin{pmatrix} D & r \\ mN & s \end{pmatrix}\cancel{\begin{pmatrix} 1 & \frac{2r}{D} \\ 0 & 1 \end{pmatrix}} z\right)$$

$$+\ \psi\left(\begin{pmatrix}1 & -\frac{2r}{s}\\ 0 & 1\end{pmatrix}\begin{pmatrix}D & r\\ mN & s\end{pmatrix}\cancel{\begin{pmatrix}1 & \frac{2r}{D}\\ 0 & 1\end{pmatrix}}\, z\right).$$

$$\to\ \psi\left(\begin{pmatrix}D & -r\\ -mN & s\end{pmatrix}\cancel{\begin{pmatrix}1 & \frac{2r}{D}\\ 0 & 1\end{pmatrix}}\, z\right) - \psi\left(\cancel{\begin{pmatrix}1 & \frac{2r}{D}\\ 0 & 1\end{pmatrix}}\, z\right)$$

$$\boxed{z \mapsto \begin{pmatrix}1 & -\frac{2r}{D}\\ 0 & 1\end{pmatrix} z.}$$

$$= -\psi\left(\underbrace{\begin{pmatrix}s & r\\ mN & D\end{pmatrix}\begin{pmatrix}1 & -\frac{2r}{s}\\ 0 & 1\end{pmatrix}\begin{pmatrix}D & r\\ mN & s\end{pmatrix}\begin{pmatrix}1 & -\frac{2r}{D}\\ 0 & 1\end{pmatrix}}\, z\right)$$

$$+\ \psi\left(\begin{pmatrix}1 & -\frac{2r}{s}\\ 0 & 1\end{pmatrix}\begin{pmatrix}D & r\\ mN & s\end{pmatrix}\begin{pmatrix}1 & -\frac{2r}{D}\\ 0 & 1\end{pmatrix} z\right).$$

$$\begin{pmatrix}D & -r\\ -mN & s\end{pmatrix}\begin{pmatrix}s & r\\ mN & D\end{pmatrix}.$$

efunct of Δ,

Let $\underbrace{\psi_1(z)} := \psi\left(\begin{pmatrix}D & -r\\ -mN & s\end{pmatrix} z\right) - \psi(z)$

$SL_2(\mathbb{R})$.

ψ

$$= \psi_1(Mz), \quad \text{where}$$

$$M = \begin{pmatrix}s & r\\ mN & D\end{pmatrix}\begin{pmatrix}1 & -\frac{2r}{s}\\ 0 & 1\end{pmatrix}\begin{pmatrix}D & r\\ mN & s\end{pmatrix}\begin{pmatrix}1 & -\frac{2r}{D}\\ 0 & 1\end{pmatrix} = \begin{pmatrix}1 & -\frac{2r}{D}\\ \frac{2mN}{s} & -3+\frac{4}{Ds}\end{pmatrix}$$

$Ds - mNr = 1, \quad mNr = Ds - 1$

$\det M = 1 \qquad \operatorname{tr} M = -2 + \frac{4}{Ds} \in (-2,2).$

$\Rightarrow M$ is elliptic.

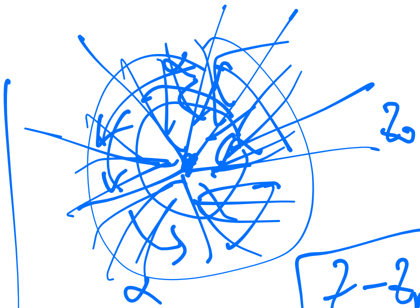
is

$\begin{pmatrix} \cos\alpha & \sin\alpha \\ -\sin\alpha & \cos\alpha \end{pmatrix} \in K = SO(2)$

Not an algebraic integer

$\Rightarrow \alpha$

not a rat mult of π.



$\boxed{\begin{aligned} z - z_0 \\ = re^{i\theta} \end{aligned}}$

Argue by taking Fourier expansion of ψ_1 about fixed point of M. $\Rightarrow$ Fourier coeff's are all 0.
(& ψ_1 is cuspidal)

$\Rightarrow \psi_1 \equiv 0. \Rightarrow \psi\left(\begin{pmatrix} D & -r \\ -mN & s \end{pmatrix} z\right) = \psi(z)$

$\forall\ D, s$ prime & $(Ds, 2N) = 1$. $\quad Ds - mNr = 1$

$Ds \equiv 1 \ (N)$.

Want: $\forall\ a, b, c, d, \quad \begin{pmatrix} a & b \\ cN & d \end{pmatrix} \in \Gamma_0(N)$:

$\psi\left(\begin{pmatrix} a & b \\ cN & d \end{pmatrix} z\right) = \psi(z). \quad \left(\Rightarrow \varphi = \psi(w_N^{-1}) \text{ also on } \Gamma_0(N)\right)$

By Fourier series of ψ, $\psi\left(\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} z\right) = \psi(z)$.

Want to write

$$\begin{pmatrix} a & b \\ cN & d \end{pmatrix} = \begin{pmatrix} 1 & u \\ 0 & 1 \end{pmatrix} \begin{pmatrix} D & -r \\ -mN & s \end{pmatrix} \begin{pmatrix} 1 & v \\ 0 & 1 \end{pmatrix}$$

Can I find such a rep of ?

$$\text{RHS} = \begin{pmatrix} D - mNu & -r + us \\ -mN & s \end{pmatrix} \begin{pmatrix} 1 & v \\ 0 & 1 \end{pmatrix}$$

$$= \begin{pmatrix} \underline{D - mNu} & (D - mNu)v + (-r + us) \\ \underline{-mN} & \underline{\underline{-mvN + s}} \end{pmatrix} \overset{?}{=} \begin{pmatrix} \underline{a} & b \\ \underline{cN} & \underline{\underline{d}} \end{pmatrix}.$$

Set $m := -c$, $D + cNu = a$?.

i.e. $\exists ?\, u$ s.t. $D := a - c N \underline{\underline{u}}$ is prime?.

Arithm prog of step $\underline{cN}$, shift $\underline{a}$; $(a, N) = 1$,

By $\underline{\text{Dirichlet's}}$ thm $\exists\, u$ s.t. $D = a - cNu$ is prime

$d = cNv + s$, i.e. $s \overset{?}{=} d - cNv$, $(d, cN) = 1$.

& need $s \equiv \bar{D} \pmod{N}$. $\overset{\text{Dirichlet}}{\Rightarrow} \exists\, s$ prime

Summary: Find u s.t. $D = a - cNu =$ prime

Find v s.t. $s = d - cNv =$ prime

& $s \equiv \bar{D}\ (N)$.

Then $\psi\left(\begin{pmatrix} a & b \\ cN & d \end{pmatrix} z\right) = \psi\left(\begin{pmatrix} 1 & u \\ 0 & 1 \end{pmatrix}\begin{pmatrix} D & * \\ cN & s \end{pmatrix}\begin{pmatrix} 1 & v \\ 0 & 1 \end{pmatrix} z\right)$