Last time: Mellin transform

$$\tilde{\varphi}(s) = \int_0^\infty \varphi(y)\, y^s \frac{dy}{y}, \qquad \text{inversion: } \varphi(y) = \frac{1}{2\pi i}\int_{(2)} \tilde{\varphi}(s)\, y^{-s}\, ds$$

$(2) \leftarrow \operatorname{Re} s = 2$

Hecke operator on automorphic $f \in L^2(\Gamma\backslash\mathbb{H})$,

$$(T_N f)(z) := \frac{1}{N^{1/2}} \sum_{M \in \Gamma\backslash\Delta_N} f(Mz)$$

$$= \frac{1}{N^{1/2}} \sum_{\alpha\cdot\delta = N} \sum_{\beta(\delta)} f\left(\frac{\alpha z+\beta}{\delta}\right).$$

$\Delta_N = \{ M_2(\mathbb{Z}) \mid \det = N \}$

$\Gamma\backslash\Delta_N = \left\{ \begin{pmatrix} \alpha & \beta \\ 0 & \delta \end{pmatrix} \;\middle|\; \alpha\cdot\delta = N,\ \beta(\delta) \right\}$

T_N's commute with each other & with $\Delta = y^2(\partial_{xx} + \partial_{yy})$

Let $\varphi \in L^2_0(\Gamma\backslash\mathbb{H})$ be a simultaneous eigenfunction, i.e.

$$\underbrace{\Delta\varphi = \lambda_\infty \varphi}, \quad \underbrace{\lambda = \tfrac{1}{4} + r^2 \geq \tfrac{1}{4}}; \quad \underbrace{T_N \varphi = \lambda_N \varphi}.$$

$\Downarrow$

$$\varphi(z) = \sum_{n \geq 1} a_n\, y^{1/2} K_{ir}(2\pi n y)\, e(nx) \qquad \text{"Hecke-Maass cusp form".}$$

$$\lambda_N \cdot \varphi = (T_N \varphi)(z) = \frac{1}{N^{1/2}} \sum_{\alpha\cdot\delta=N} \sum_{\beta(\delta)} \sum_{n\geq 1} a_n \left(\frac{\alpha}{\delta} y\right)^{1/2} K_{ir}\left(2\pi n \frac{\alpha}{\delta} y\right) e\left(n \frac{\alpha x+\beta}{\delta}\right)$$

$$\sum_{\beta(\delta)} e\left(n \frac{\alpha x+\beta}{\delta}\right) \;\to\; \begin{cases} \delta, & n \equiv 0 (\delta) \\ 0, & \text{else} \end{cases}$$

$$= \frac{1}{\cancel{N^{1/2}}} \sum_{\alpha\cdot\delta=N} \sum_{\substack{\cancel{n\equiv 0(\delta)} \\ n \geq 1}} \cancel{\delta} \cdot a_{n\cdot\delta} \cancel{\left(\frac{\alpha}{\delta}\right)^{1/2}} y^{1/2} K_{ir}\left(2\pi \frac{n\alpha}{\delta} y\right) e\left(n \frac{\alpha x}{\delta}\right)$$

$n \mapsto n\cdot\delta$ Let $m = n\cdot\alpha$

$$\delta \cdot n = \frac{m}{\alpha} \cdot \frac{N}{\alpha}$$

$$= \sum_{m\geq 1} \boxed{\sum_{\substack{\alpha\cdot\delta=N \\ \alpha \mid m}} a_{\frac{m\cdot N}{\alpha^2}}} \cdot y^{1/2} K_{ir}(2\pi m y)\, e(mx).$$

$$= \sum_{m\geq 1} \boxed{\lambda_N \cdot a_m} \cdot y^{1/2} K_{ir}(2\pi m y)\, e(mx).$$

Lemma:

$$\lambda_N \cdot a_m = \sum_{\alpha \mid (N,m)} a_{\frac{m\cdot N}{\alpha^2}}.$$

When $m=1$: $\lambda_N \cdot a_1 = a_N$

$\Rightarrow \alpha = 1$

$$\boxed{a_p \cdot a_{p^\ell} = \underbrace{a_{p^{\ell+1}}}_{\alpha=1} + \underbrace{a_{p^{\ell-1}}}_{\alpha=p}}$$

$\Rightarrow$ If $a_1 = 0 \Rightarrow a_N = 0 \ \forall N, \Rightarrow \varphi = 0.$

If $a_1 \neq 0$, normalize φ so that $a_1 = 1. \Rightarrow \lambda_N = a_N$!!!

eigenvalues $\to \lambda_N$

Fourier coeffs!! $\to a_N$.

If $(m,N)=1$, $\Rightarrow \alpha = 1$,

$a_N \cdot a_m = a_{N \cdot m}$ multiplicative!

$$\sum \frac{\lambda_N}{N^s} = \sum \frac{a_N}{N^s}.$$

$$\Rightarrow \sum_{n \geq 1} \frac{a_n}{n^s} = \prod_p \left(1 + \frac{a_p}{p^s} + \frac{a_{p^2}}{p^{2s}} + \frac{a_{p^3}}{p^{3s}} + \frac{a_{p^4}}{p^{4s}} + \cdots \right).$$

$$S = 1 + a_p \cdot X + a_{p^2} \cdot X^2 + a_{p^3} \cdot X^3 + \cdots \quad \left(X = \frac{1}{p^s}\right).$$

$$1 + a_p \cdot X \cdot S = a_p \cdot X \left(1 + a_p \cdot X + a_{p^2} \cdot X^2 + a_{p^3} \cdot X^3 + \cdots \right)$$

$$= 1 + a_p X + a_{p^2} X^2 + \overset{1}{\cancel{a_1}} \cdot X^2 + a_{p^3} X^3 + a_p \cdot X^3 + a_{p^4} X^4 + a_{p^2} X^4 + \cdots$$

$$= S + X^2 \cdot S, \Rightarrow S = (1 - a_p X + X^2)^{-1}$$

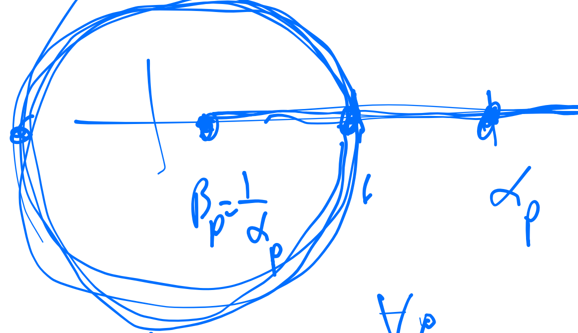
Langlands-Satake param…

$$\Rightarrow \sum_{n \geq 1} \frac{a_n}{n^s} = L(\varphi, s) = \prod_p \left(1 - \frac{a_p}{p^s} + \frac{1}{p^{2s}}\right)^{-1} = \prod_p \left(1 - \frac{\alpha_p}{p^s}\right)^{-1} \left(1 - \frac{\beta_p}{p^s}\right)^{-1}$$

where $\alpha_p \cdot \beta_p = 1$, $\alpha_p + \beta_p = a_p \in \mathbb{R}. \Rightarrow$ $(\mathrm{Re}\, s > 10)$.

either α_p, β_p both real,

or $\alpha_p = \overline{\beta_p}$, $|\alpha_p|^2 = 1$



"Ramanujan Conj": truth not this $\Rightarrow |a_p| \le 2$ $\forall p$.

"trivial" bound: $\underline{a_n \le C \cdot n^{1/2}}$. Best known (Kim-Sarnak '03): $|a_p| \le p^{7/64}$.

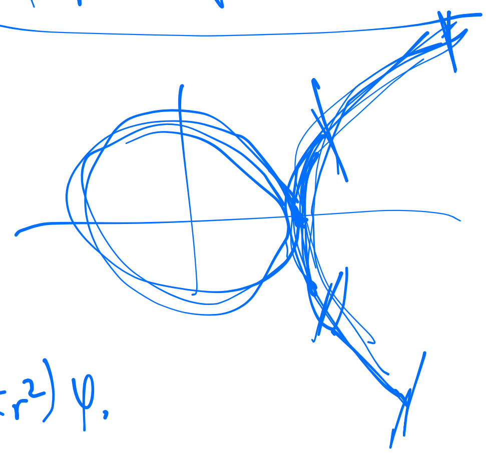
$J = x + y$, $x \cdot y = 1$.

$= x + \frac{1}{x}$, $1.5 = x + \frac{1}{x}$.

$= 2\cosh(\log x)$

$2\cos(i\log x)$.

$\boxed{\cosh x = \cos ix}$



$\Delta\varphi = \left(\frac{1}{4} + r^2\right)\varphi$.

$\forall y > 0$: $\left|\int_0^1 \varphi(x+iy)\, e(-nx)\, dx\right| = \left|a_n\, y^{1/2} K_{ir}(2\pi n y)\right| \le C$.

$|\varphi| \le C$ on all $z \in \mathbb{H}$.

$\Rightarrow |a_n| \le C \cdot y^{-1/2} \frac{1}{K_{ir}(2\pi n y)}$.

decays exp. So $\frac{1}{K}$ grows exp in y!! If set $y = 1$, $|a_n| \le C \cdot e^{cn}$,

then $L(\varphi, s)$ converges nowhere!! $\sum \frac{a_n}{n^s}$.

Key idea: $y = \frac{1}{n} \Rightarrow |a_n| \le C \cdot n^{1/2} \frac{1}{K_{ir}(2\pi)} \le C_\varphi \cdot n^{1/2}$.

∞-analogue of Ramanujan conj is Selberg $\frac{1}{4}$-conj:

Let Γ be a congruence gp, that is, $\Gamma > \Gamma(q)$ = principal congruence group of level q = $\ker SL_2(\mathbb{Z}) \twoheadrightarrow SL_2(\mathbb{Z}/q)$

$= \{\gamma \equiv \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \pmod q, \gamma \in SL_2(\mathbb{Z})\}$

$\Gamma_1(q) = \{\gamma = \begin{pmatrix} 1 & * \\ 0 & 1 \end{pmatrix} (q)\}$,

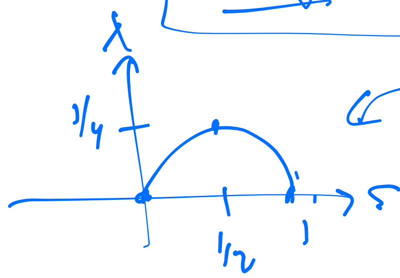
$\Gamma_0(q) = \{\gamma = \begin{pmatrix} * & * \\ 0 & * \end{pmatrix} (q)\}$.

If $\varphi \in L^2(\Gamma \backslash \mathbb{H})$

$\Delta\varphi = \lambda\varphi$, $\varphi \not\equiv c$.

$\boxed{\text{Conj: } \lambda_1 \geq 1/4.}$

$\lambda = s(1-s) \geq 0$.



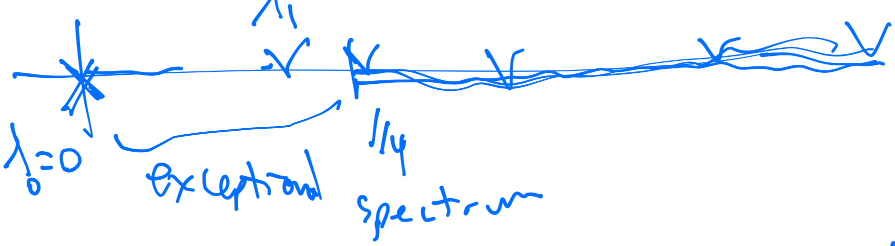
~~If $s \in \mathbb{R}$~~,

OR

$s = \frac{1}{2} + ir$,

$1 - s = \frac{1}{2} - ir$, $\quad s(1-s) = \frac{1}{4} + r^2$.

Spec Δ



Selberg: $\exists$ subgroups of $SL_2(\mathbb{Z})$ of finite index s.t. $\lambda_1 \to 0$.
(non-congruence).

Aside: Margulis Arithmeticity: If G is a higher rank group & Γ lattice in $G(\mathbb{R}) \Rightarrow \Gamma$ arithmetic.

Thm: Selberg 1/4 holds for $\Gamma = SL_2(\mathbb{Z})$.

I.e. φ Maass cusp form for $SL_2(\mathbb{Z}) \Rightarrow \boxed{\lambda \geq 1/4}$.

Pf: $$\lambda \cdot \|\varphi\|^2 = \langle \underbrace{\Delta\varphi}_{\lambda_\varphi}, \varphi \rangle = \int \left(-\cancel{y^2}(\partial_{xx}+\partial_{yy})\varphi\right) \cdot \bar{\varphi}\, \frac{dx\,dy}{\cancel{y^2}}$$

$\Gamma\backslash\mathbb{H} : D$

$\Rightarrow \boxed{\lambda \geq 0}$.

integrate by parts $$= + \int_D |(\partial_x+\partial_y)\varphi|^2\, dx\,dy \; \boxed{\geq 0}.$$

D. $S = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$.

$$\geq \frac{1}{2}\int_{\sqrt{3}/2}^{\infty}\left[\int_{-1/2}^{1/2} |\cancel{y}(\partial_x+\partial_y)\varphi|^2\, dx\right]\frac{dy}{y^2}$$

$$\int_0^1 |(\partial_x+\partial_y)\varphi(x+iy)|^2\, dx = \int_0^1 \left|\sum_{n\geq 1} a_n \left[y^{1/2}K_{ir}(2\pi n y)\, 2\pi i n\, e(nx) + \frac{\partial}{\partial y}\left[y^{1/2}K_{ir}(2\pi n y)\right]\right]\right|^2 dx$$

Parseval

$$\geq \sum_{n\geq 1} |a_n|^2 \left[\cancel{\left(\frac{\partial}{\partial y}\left[y^{1/2}K_{ir}(2\pi n y)\right]\right)^2} + 4\pi^2 \cancel{n^2}\left(y^{1/2}K_{ir}(2\pi n y)\right)^2\right]$$

$$\geq \frac{1}{2}\cdot 4\pi^2 \int_{\sqrt{3}/2}^{\infty} \sum_{n\geq 1} |a_n|^2 \left(y^{1/2}K_{ir}(2\pi n y)\right)^2 \frac{dx\,dy}{y^2} \ldots$$

$\boxed{\text{TBC}}$