Last time: Recall: $GL_1(\mathbb{Q})\backslash GL_1(\mathbb{A}) = \mathbb{Q}^\times\backslash\mathbb{A}^\times$

$$= \underbrace{GL_1(\mathbb{Z})\backslash GL_1(\mathbb{R})}_{(0,\infty) = \mathbb{Z}^\times\backslash\mathbb{R}^\times} \times \prod_p \underbrace{GL_1(\mathbb{Z}_p)}_{= \mathbb{Z}_p^\times}.$$

Now: $\iota_\Delta(GL_2(\mathbb{Q}))\backslash GL_2(\mathbb{A}) = ?$,

Recall: given any $g \in GL_2(\mathbb{A})$, $\exists q \in \mathbb{Q}^\times$ $\exists x \in (0,\infty)$ $\exists u \in \prod_p \mathbb{Z}_p^\times$ s.t. $\det g = \iota_\Delta(q)\cdot\iota_\infty(x)\iota_f(u)$

$$g_1 = \iota_\Delta\begin{pmatrix} q^{-1} & 0 \\ 0 & 1 \end{pmatrix} g \underbrace{\left(\begin{pmatrix} x^{-1} & 0 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} u_2^{-1} & 0 \\ 0 & 1 \end{pmatrix}, \ldots\right)}_{\iota_\infty\begin{pmatrix} x^{-1} & 0 \\ 0 & 1 \end{pmatrix}\cdot\iota_f\begin{pmatrix} u^{-1} & 0 \\ 0 & 1 \end{pmatrix}} \in SL_2(\mathbb{A}).$$

Strong Approx: $\iota_\Delta(SL_2(\mathbb{Q}))\cdot\iota_\infty(SL_2(\mathbb{R}))$ dense in $SL_2(\mathbb{A})$. (CRT).

$$g_1 \cdot (\underset{SL_2(\mathbb{R})}{\varepsilon_{\infty,1}}, \underset{SL_2(\mathbb{Z}_p)}{\varepsilon_{2,1}}, \dots) = i_\Delta(\underset{SL_2(\mathbb{Q})}{\gamma})\, i_\infty(\underset{SL_2(\mathbb{R})}{n})$$

$$g = \underbrace{i_\Delta\begin{pmatrix} q & 0 \\ 0 & 1\end{pmatrix}}_{} \cdot \underbrace{i_\Delta(\gamma)}_{}\, i_\infty(n)\, \underbrace{i_\infty(\varepsilon_\infty^{-1})}_{i_\infty\begin{pmatrix} x & 0 \\ 0 & 1\end{pmatrix}}\, \underbrace{i_f(\underset{SL_2(\mathbb{Z}_p)}{\varepsilon_p^{-1}})}_{i_f\begin{pmatrix} u & 0 \\ 0 & 1\end{pmatrix}}$$

$$g \in i_\Delta GL_2(\mathbb{Q}) \cdot GL_2(\mathbb{R}) \times \prod_p GL_2(\mathbb{Z}_p).$$

Summary: $\forall g \in GL_2(\mathbb{A})$, $\exists \gamma \in GL_2(\mathbb{Q})$

s.t. $i_\Delta(\gamma)^{-1} \cdot g \in GL_2(\mathbb{R}) \times \prod_p GL_2(\mathbb{Z}_p)$.

So: $GL_2(\mathbb{Q}) \backslash GL_2(\mathbb{A}) \overset{?}{=} \underbrace{GL_2(\mathbb{R}) \times \prod_p GL_2(\mathbb{Z}_p)}_{\ni n}$

Is rep'n of $g = i_\Delta(\gamma) \cdot n$ unique?

If another: $g = i_\Delta(\gamma') \cdot n'$, then

$i_\Delta(\gamma^{-1}\gamma') \quad n\, n'^{-1} \in GL_2(\mathbb{R}) \prod GL_2(\mathbb{Z}_p)$

$$L_\Delta(\underbrace{\gamma\cdot\gamma'}) = M\cdot(M') \in GL_2(\mathbb{R})\times\prod_p GL_2(\mathbb{Z}_p) \cap L_\Delta(GL_2(\mathbb{Q})),$$

$\Rightarrow$ has no denominators!

$$\Rightarrow \quad \gamma^{-1}\cdot\gamma' \in GL_2(\mathbb{Z}).$$

__Thm:__ A fnd domain for $GL_2(\mathbb{Q})\backslash GL_2(\mathbb{A})$ is:

$$L_\infty\left(\underbrace{GL_2(\mathbb{Z})\backslash GL_2(\mathbb{R})}_{\text{not fin vol}}\right)\times\prod_p GL_2(\mathbb{Z}_p).$$

__Compare:__ $GL_1(\mathbb{Q})\backslash GL_1(\mathbb{A}) = \underbrace{GL_1(\mathbb{Z})\backslash GL_1(\mathbb{R})}_{(0,\infty)\ \underline{\text{not}}\ \text{fin vol}} \times \prod_p GL_1(\mathbb{Z}_p)$

__Alternate version:__ $\underbrace{Z(\mathbb{A})GL_2(\mathbb{Q})\backslash GL_2(\mathbb{A})}_{\text{adelic Aut form}} \Big/ O(2)\times\prod_p GL_2(\mathbb{Z}_p)$

$Z(\mathbb{A})$ = Center = $\left\{\begin{pmatrix} a & 0\\ 0 & a\end{pmatrix}, a\in\mathbb{A}^\times\right\}$.

$$= SL_2(\mathbb{Z})\backslash SL_2(\mathbb{R})/SO(2)$$

$$= \underbrace{SL_2(\mathbb{Z})\backslash \mathbb{H}}_{\text{fin. volume}}.$$

$$\varphi : Z(\mathbb{A})GL_2(\mathbb{Q}) \backslash GL_2(\mathbb{A}) \to \mathbb{C},$$

$$\varphi(\gamma g) = \varphi(g) \quad \forall \gamma \in GL_2(\mathbb{Q}),\ g \in GL_2(\mathbb{A}),$$

& $\underbrace{\text{eigenfunction}}$ $\left(\text{e.g. } \Delta = y^2(\partial_{xx}+\partial_{yy}),\ \partial_{\bar z} = 0 \right)$ $\underbrace{\text{grow rates}}$ $\in L^2(\Gamma\backslash G)$,

TBC...

(classical) L-functions.

$$\Delta(z) = q \prod_{n\geq 1}(1-q^n)^{24}$$

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level 1

modular cusp form

$q = e^{2\pi i z}$

$$= q - 24q^2 + \cdots$$

$$= \sum_{n\geq 1} \tau(n) q^n$$

Deligne: $|\tau(n)| \ll_\varepsilon n^{\frac{11}{2}+\varepsilon}$.

$$\Delta(\gamma z) = (cz+d)^{12}\Delta(z)$$

$\forall \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL_2(\mathbb{Z})$, $\forall z \in \mathbb{H}$.

Want: attach L-function to Δ?

Convert Fourier expansion into Dirichlet series.

Hecke: 

$$\boxed{\int_0^\infty \Delta(iy) \cdot y^{\frac{11}{2}} \cdot y^s \frac{dy}{y}}$$

(dx, dy)

$$g_z\left(\begin{smallmatrix}'' \\ \xi\end{smallmatrix}\right) = \frac{dx^2+dy^2}{y^2}.$$

(Mellin transform!).

(modularity $\leftrightsquigarrow$ Poisson sum).

$$\int_0^\infty \underbrace{\sum_{n\geq 1} \tau(n)\, e^{-2\pi n y}}\cdot y^{\frac{11}{2}} \cdot y^s \frac{dy}{y}.$$

(Re $s>1$).

$y \mapsto \dfrac{y}{2\pi n}$

$$= (2\pi)^{-(s+\frac{11}{2})} \underbrace{\sum_{n\geq 1} \frac{\tau(n)/n^{11/2}}{n^s}} \underbrace{\int_0^\infty e^{-y}\, y^{\frac{11}{2}}\, y^s \frac{dy}{y}} = \Lambda(\Delta, s)$$

$$\cdot L(\Delta, s) \qquad \cdot \Gamma\left(s+\tfrac{11}{2}\right). \qquad = \Lambda(\Delta, 1-s).$$

Note: $\Delta\left(\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} iy\right) = \Delta\left(\frac{i}{y}\right) = (iy+0)^{12}\Delta(iy)$

($\Rightarrow$ have exp decay both as $y \to \infty$ & as $y \to 0$).

$y \mapsto \frac{1}{y}$

$$\Rightarrow \int_0^\infty \Delta\left(\frac{i}{y}\right) y^{-11/2}\, y^{-s} \frac{dy}{y}$$

Euler product?

TBC...

$$= \int_0^\infty \quad (\quad)^{12} \quad y^{-11} \quad y^{-s} \quad d$$

$$\int_0 \Delta(iy)\, \underbrace{y^{1/2} \cdot y^{-1/2}\, y}\, \frac{dy}{y}$$

Hecke op's.

$$y^{1/2} \cdot y^{1-s}$$

How to get L-functions from Maass forms?

Suppose $f : SL_2(\mathbb{Z}) \backslash \mathbb{H} \to \mathbb{C}$, eigenfunction of $\Delta = -y^2(\partial_{xx} + \partial_{yy})$,

& $f \in L^2(\Gamma \backslash \mathbb{H})$. (Q: Do these eigen exist?...). Yes ....

$$f(1+x+iy) = f(x+iy) = \sum_{n \in \mathbb{Z}} \underline{\underline{a_n(y)}}\, e^{2\pi i n x}.$$

$$\lambda f = \Delta f = \sum_{n \in \mathbb{Z}} \underbrace{-y^2\left(-4\pi^2 n^2 + a_n''(y)\right)}\, e^{2\pi i n x},$$

$$= \sum_{n \in \mathbb{Z}} \underbrace{\lambda \cdot a_n(y)}\, e^{2\pi i n x}$$

System of 2nd order ODE on $a_n(y)$ $(n \neq 0)$.

Turns out: 2 indep solutions I- & K-Bessel functions.

Gauss hypergeometric

$${}_2F_1(a,b,c;z) = \sum_{n\ge 0} \frac{(a)_n (b)_n}{(c)_n} \frac{z^n}{n!} \qquad (a)_n = a(a+1)\cdots(a+n).$$

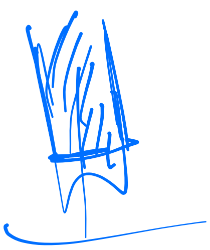
K-Bessel has exp decay as $y \to \infty$.

I-Bessel has exp growth as $y \to \infty$.

$\lambda = s(1-s)$.

So, $$f(x+iy) = \sum_{n\in\mathbb{Z}} \Big[a_n \cdot y^{1/2} K_{s-1/2}(2\pi n y) + b_n \cdot y^{1/2} I_{s-1/2}(2\pi n y)\Big] e^{2\pi i n x}$$

$\in L^2(\Gamma \backslash \mathbb{H})$.



$$\infty > \int_{\mathcal{F}} |f|^2 > \int_{10}^{\infty}\int_{-1/2}^{1/2} |f|^2 \, dx \, \frac{dy}{y^2}$$

$$= \int_{10}^{\infty} \sum_{n\in\mathbb{Z}} \left| a_n y^{1/2} K + b_n y^{1/2} I \right|^2 \frac{dy}{y^2}.$$

$\Rightarrow$ all δ_n's are 0. $\Rightarrow$ Mult one.

$$\Rightarrow f(x+iy) = \sum_{n \in \mathbb{Z}} a_n \cdot y^{1/2} K_{s-1/2}(2\pi n y) e^{2\pi i n x}$$

If f is a cusp form $\Leftrightarrow$ $\int_0^1 f(x+iy)\,dx = 0$,

$a_0 = 0$. $\Leftarrow$

Then you can make an L-function...