Very roughly: $\to L(\pi, s)$ has FE & Euler product & growth rates.

$\to$ if L has deg n. & "$\forall$" deg $\leq n-2$ ρ, ($\leq n-1$)

$\to L(\pi \otimes \rho, s)$ also has FE & Euler prod.

Then $\to \exists\, \tilde{\pi}$ aut rep V, s.t. $L(\tilde{\pi}, s) = L(\pi, s)$.

$\varphi \in V$.

Very roughly $L(s) = \sum \frac{a_n}{n^s}$, φ "=" $\sum a_n q^n$. aut form

Future TOU: Converse Thm: is φ modular?

Last time: Thm (Iwasawa decomp for $GL_2(\mathbb{Q}_p)$):

$\forall g \in GL_2(\mathbb{Q}_p)$, $\exists !$ $g = \begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix} \begin{pmatrix} p^{e_1} & \\ & p^{e_2} \end{pmatrix} \cdot k$. ($x \in \mathbb{Q}_p$)

$e_1, e_2 \in \mathbb{Z}$, $k \in K_p = GL_2(\mathbb{Z}_p)$. & $x = a_{-N} p^{-N} + \cdots$ has $a(n) = 0\ \forall n \geq e_1 - e_2$.

Pf: $gk_1 = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \underbrace{\begin{pmatrix} p^\ell d & * \\ p^\ell(-c) & * \end{pmatrix}}_{k_1} = \begin{pmatrix} t_1 & x \\ 0 & t_2 \end{pmatrix}$ Let $p^\ell = \max(|c|_p, |d|_p)$

$$t_1 = p^{e_1} \cdot u_1, \quad t_2 = p^{e_2} \cdot u_2$$

$$u_1 \in \mathbb{Z}_p^\times$$

$$g \cdot \underbrace{k_1 \begin{pmatrix} u_1^{-1} & \\ & u_2^{-1} \end{pmatrix}}_{\in K_p} \begin{pmatrix} p^{-e_1} & \\ & p^{-e_2} \end{pmatrix} = \begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix}$$

$$\Rightarrow \exists \quad g = \begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix} \begin{pmatrix} p^{e_1} & \\ & p^{e_2} \end{pmatrix} k, \quad k \in K_p.$$

Is this unique?

<u>Ans: No!</u> If $g = \begin{pmatrix} 1 & x' \\ 0 & 1 \end{pmatrix} \begin{pmatrix} p^{e_1'} & \\ & p^{e_2'} \end{pmatrix} k'$.

$$\begin{pmatrix} p^{-e_1'} & \\ & p^{-e_2'} \end{pmatrix} \underbrace{\begin{pmatrix} 1 & -x' \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix}} \begin{pmatrix} p^{e_1} & \\ & p^{e_2} \end{pmatrix} = k' \cdot k^{-1} \in \underbrace{GL_2(\mathbb{Z}_p)}$$

$$\begin{pmatrix} p^{-e_1'} & 0 \\ & \end{pmatrix} \begin{pmatrix} 1 & x-x' \\ 0 & 1 \end{pmatrix} \begin{pmatrix} & \\ & p^{e_2} \end{pmatrix}$$

$$\begin{pmatrix} p^{e_1-e_1'} & p^{e_2-e_1'}(x-x') \\ 0 & p^{e_2-e_2'} \end{pmatrix}$$

$e_1 - e_1' \geq 0.$

$e_2 - e_2' \geq 0,$

$e_1 - e_1' + e_2 - e_2' = 0.$

$|\det g|_p = 1.$

$$\Rightarrow \underline{e_1 = e_1', \quad e_2 = e_2'}.$$

$$\begin{pmatrix} 1 & p^{e_2-e_1}(x-x') \end{pmatrix} \in GL_2(\mathbb{Z}_p)$$

$$\begin{pmatrix} \\ 0 & 1 \end{pmatrix} \qquad \dots e_2 \Big) \; e_p \Big)$$

$$\iff \underbrace{p^{e_2-e_1}(x-x')} \in \mathbb{Z}_p \xrightarrow{\text{Want}} x = x'.$$

$$x = a_{-N}p^{-N} + \dots + a_{-1}p^{-1} + a_0 + a_1 p + \dots$$

Need: Restrict x to $x = \underbrace{a_{-N}p^{-N} + \dots + a_{e_1-e_2+1} \cdot p^{e_1-e_2-1}}$

i.e. $\forall\, n \geq e_1-e_2+1$, $a_n(x) = 0$.

Then $x' = a'_{-N'}p^{-N'} + \dots + a'_{e_1-e_2-1}\, p^{e_1-e_2-1}$

Then $\left| p^{e_2-e_1}(x-x') \right|_p \geq p$ unless all coeff's agree,

i.e. $x = x'$.

Write $x = \underbrace{a_{-N}p^{-N} + \dots + a_{e_1-e_2-1}\, p^{e_1-e_2-1}}_{x_0} + p^{e_1-e_2}u$, $u \in \mathbb{Z}_p$.

We have written

$$g = \begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix} \begin{pmatrix} p^{e_1} & \\ & p^{e_2} \end{pmatrix} K,$$

$$= \begin{pmatrix} 1 & x_0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & p^{e_1-e_2}u \\ 0 & 1 \end{pmatrix} \begin{pmatrix} p^{e_1} & \\ & p^{e_2} \end{pmatrix} K.$$

$$= \begin{pmatrix}1 & x_p\\ 0 & 1\end{pmatrix}\begin{pmatrix}p^{e_1} & \\ & p^{e_2}\end{pmatrix}\underbrace{\begin{pmatrix}p^{-e_1} & \\ & p^{-e_2}\end{pmatrix}\begin{pmatrix}1 & p^{e_1-e_2}a\\ 0 & 1\end{pmatrix}\begin{pmatrix}p^{e_1} & \\ & p^{e_2}\end{pmatrix}k}\,,$$

$$= \begin{pmatrix}1 & x_p\\ 0 & 1\end{pmatrix}\begin{pmatrix}p^{e_1} & \\ & p^{e_2}\end{pmatrix}\underbrace{\begin{pmatrix}1 & u\\ 0 & 1\end{pmatrix}\cdot k}_{\in K_p}.$$

Def: $GL_2(\mathbb{A}) = GL_2(\mathbb{R}) \times \prod_p{}' GL_2(\mathbb{Q}_p)$.

$$= \left\{ (g_\infty, g_2, g_3, \dots),\ g_p \in GL_2(\mathbb{Q}_p)\ \forall p,\ g_p \in GL_2(\mathbb{Z}_p) \text{ a.a. } p \right\}$$

Thm $\left\{ \begin{pmatrix}a & b\\ c & d\end{pmatrix},\ a, b, c, d \in \mathbb{A},\ \underbrace{ad-bc} \in \mathbb{A}^\times \right\}$.

Thm (adelic Iwasawa decomp): $\forall\, g \in GL_2(\mathbb{A})$,

$$\exists!\quad g = \begin{pmatrix}1 & x\\ 0 & 1\end{pmatrix}\begin{pmatrix}y & 0\\ 0 & 1\end{pmatrix}\begin{pmatrix}r & 0\\ 0 & r\end{pmatrix}\cdot k,$$

Where: $x=(x_\infty, x_2, \dots)\in \mathbb{A}$, $y=(y_p)\in\mathbb{A}^\times$, $r\in\mathbb{A}^\times$,

$K = K_{\mathbb{A}} = O(2)\times\prod_p GL_2(\mathbb{Z}_p)$ & $r_p = p^{e_2(p)}$, $r_\infty>0$

$e_2(p)=0$ a.a. p, $y_\infty>0$, $y_p = p^{e_1(p)-e_2(p)}$, $e_1(p)=0$ a.a. p,

& $x_\infty\in\mathbb{R}$, $x_p = \sum_{\ell=-N_p}^{e_1(p)-e_2(p)-1} a_\ell(p)\cdot p^\ell$. pf: local $\to$ global.

Recall: $GL_1(\mathbb{Q})\backslash GL_1(\mathbb{A}) = (0,\infty)\times\prod_p \mathbb{Z}_p^\times$.

$\mathbb{Z}^\times\backslash\mathbb{R}^\times$

Want: Find dom for $GL_2(\mathbb{Q})\backslash GL_2(\mathbb{A})$?

I.e. given $g\in GL_2(\mathbb{A})$, how "simple" can we make it by left-mult by $GL_2(\mathbb{Q})$?

Look at $\det g\in\mathbb{A}^\times$.

$\Rightarrow \exists\ q\in\mathbb{Q}^\times$, $x\in(0,\infty)$ s.t. $\det g = q\cdot x\cdot u$

$$\det g = i_\Delta(q) \cdot i_\infty(x) \cdot u, \quad i_\Delta(q) = (q, q, q, \dots), \quad i_\infty(x) = (x, 1, 1, \dots, 1), \quad u = (1, \mathbb{Z}_p^\times).$$

i.e. $$g = \underbrace{\begin{pmatrix} q^{-1} & 0 \\ 0 & 1 \end{pmatrix}}_{GL_2(\mathbb{Q})} g_1 \, i_\infty\begin{pmatrix} x^{-1} & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} u^{-1} & \\ & 1 \end{pmatrix}$$

Then $g_1 \in SL_2(\mathbb{A})$.

Thm (Strong Approx): $i_\Delta(SL_2(\mathbb{Q})) \cdot i_\infty(SL_2(\mathbb{R}))$ is dense in $SL_2(\mathbb{A})$.

pf: Need to show: $\forall$ open $U = \prod_{p \in S} U_p \times \prod_{p \notin S} SL_2(\mathbb{Z}_p)$,

where $U_p \subset SL_2(\mathbb{Q}_p)$ open, $\exists\, \gamma \in SL_2(\mathbb{Q})$, & $M \in SL_2(\mathbb{R})$

s.t. $i_\Delta(\gamma) \cdot i_\infty(M) \in U$.

Each $U_p \supset \gamma_p + p^{l_p} M_2(\mathbb{Z}_p)$.

$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL_2(\mathbb{Q}_p)$ $\qquad SL_2(\mathbb{Q}_p)$

So to find such γ, need to solve

finite

system: $\{ \gamma \equiv \gamma_p \mod p^{\ell_p} \}_{p \in S}$.

CRT ✓.

Explicit eg. $\begin{pmatrix} 2 & 0 \\ 0 & 3 \end{pmatrix} \in SL_2(\mathbb{Z}/5\mathbb{Z})$.

yes

$\exists ?\ \gamma_0 = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL_2(\mathbb{Z})$ s.t. $\gamma \equiv \begin{pmatrix} 2 & 0 \\ 0 & 3 \end{pmatrix} (5)$.

$\begin{pmatrix} 2 & 5 \\ x & 3 \end{pmatrix}$ $\qquad a = 2 + 5a_1 = -3 \qquad 1 = ad - bc$

$b = 5b_1 = -5 \qquad = 6 + 15a_1 + 10d_1 + 25(a_1d_1 - b_1c_1)$

$c = 5c_1 = 5$

$d = 3 + 5d_1 = 8 \qquad -1 = 3a_1 + 2d_1 + 5(\ \_\_\ )$

$\begin{pmatrix} -3 & -5 \\ 5 & 8 \end{pmatrix}$ $\qquad a_1 = -1, \quad d_1 = 1$

Eg. $\begin{pmatrix} 3 & 0 \\ 0 & 4 \end{pmatrix} \in GL_2(\mathbb{Z}/5\mathbb{Z})$ $\exists\ \gamma \in GL_2(\mathbb{Z})$

det = 2.

s.t. $\gamma \equiv \begin{pmatrix} 3 & 0 \\ 0 & 4 \end{pmatrix} \mod 5$?

NO!! $\det \gamma = \pm 1$.

Recall: $g_1 = \begin{pmatrix} q & 0 \\ 0 & 1 \end{pmatrix} g \begin{pmatrix} x & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} u & 0 \\ 0 & 1 \end{pmatrix}$

$\det g_1 = (1, 1, 1, \dots)$, $g_1 \in SL_2(\mathbb{A})$.

Let U_∞ be a open set in $SL_2(\mathbb{R})$ about I_2.

Look $g_1 \cdot \left[U_\infty \times \prod_p SL_2(\mathbb{Z}_p) \right] \ni \iota_\Delta(\underset{SL_2(\mathbb{Q})}{\gamma}) \cdot \iota_\infty(\underset{SL_2(\mathbb{R})}{u})$

i.e. $g_1 \cdot (\varepsilon_\infty, \varepsilon_2, \varepsilon_3, \dots) = \iota_\Delta(\gamma) \cdot \iota_\infty(u)$.

$\Downarrow$

$$\iota_\Delta \begin{pmatrix} q^{-1} & 0 \\ 0 & 1 \end{pmatrix} \underline{\underline{g}} \, \iota_\infty \begin{pmatrix} x^{-1} & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} u^{-1} & 0 \\ 0 & 1 \end{pmatrix} \varepsilon = \iota_\Delta(\gamma) \, \iota_\infty(u)$$

$$g = \underbrace{\iota_\Delta \begin{pmatrix} q & 0 \\ 0 & 1 \end{pmatrix} \iota_\Delta(\gamma)}_{\in \, GL_2(\mathbb{Q})} \underbrace{\iota_\infty(u) \begin{pmatrix} u & 0 \\ 0 & 1 \end{pmatrix} \iota_\infty \begin{pmatrix} x & 0 \\ 0 & 1 \end{pmatrix} \cdot (\varepsilon_\infty^{-1}, \varepsilon_2^{-1}, \dots)}_{\in \, GL_2(\mathbb{R}) \times \prod_p GL_2(\mathbb{Z}_p)}.$$

Summary: $\forall g \in GL_2(\mathbb{A})$, $\exists \gamma \in GL_2(\mathbb{Q})$

s.t. $i_\Delta(\gamma)^{-1} \cdot g \in GL_2(\mathbb{R}) \times \prod_p GL_2(\mathbb{Z}_p)$.

Is it unique? No . . .