Last time: Solved Diophantus's eq (300 AD.) $x^2+x^4+x^8=y^2$? $/\mathbb{Q}$.

via $\mathbb{Q}_p = \{ a_{-N}p^{-N} + a_{-N+1}p^{-N+1} + \cdots + a_{-1}p^{-1} + a_0 \cdot p^0 + a_{+1}p^1 + a_2 p^2 + \cdots \}$, $a_i \in \{0, \ldots, p-1\}$

(Hensel lifting)

"metric" Completion of $\mathbb{Q}$ w.r.t. norm $d(x,y) = |x-y|_p$, $\leftarrow$ p-adic abs val.

If $n \in \mathbb{N}$, $|n|_p = \left(\frac{1}{p}\right)^{e_p(n)} \leftarrow$ # times p divides n.

$\sim \lambda^{e_p(n)}$ for any $\lambda < 1$.

Proved Ostrowski's Thm: (1916? 1920?) Only nontrivial metric completions of $\mathbb{Q}$ are $\mathbb{Q}_p$ & $\mathbb{R}$.

Aside: $\mathbb{R}$ almost algebraically complete,

$\overline{\mathbb{R}}$ = alg closure of $\mathbb{R}$ = $\mathbb{R}[x]/(x^2+1)$

F-algebraic # / F (field/ring) α is F-alg if $\exists p \in F[x]$ s.t. $p(\alpha)=0$.

Fund Thm of Algebra: $\overline{\mathbb{R}} (= \mathbb{C})$ is alg complete,

i.e. any poly / $\mathbb{C}$ has solutions in $\mathbb{C}$,

& $\mathbb{C}$ is metrically complete.

Exercise: For which p is x^2+1 solvable in $\mathbb{Q}_p$?

Fact: $\overline{\mathbb{Q}}_p$ is an ∞ extension of $\mathbb{Q}_p$,

$\boxed{|\cdot|_p \text{ extends}}$ & $\overline{\mathbb{Q}}_p$ is not metrically complete.

$$
\begin{tikzcd}
\mathbb{R} \arrow[r, "\text{alg}"] & \mathbb{C} & \overline{\mathbb{Q}} & & \\
\mathbb{Q} \arrow[u, "\text{m.c.}"] \arrow[ur, "\text{a.c.}"] \arrow[r, "\text{m.c.}"] & \mathbb{Q}_p \arrow[r, "\text{a.c.}"] & \overline{\mathbb{Q}}_p \arrow[r, "\text{m.c.}"] & \mathbb{C}_p &
\end{tikzcd}
$$

$\longrightarrow$ metric Topology on $\mathbb{Q}_p$ metric $d(x,y) = |x-y|_p$.

Ex: $a = \frac{17}{5}$, $b = \frac{3}{20}$, $c = \frac{35}{9}$ in $\mathbb{Q}_5$.

$|a-b|_{\mathbb{Q}_5}$ $\quad |a-c|_{\mathbb{Q}_5}$ $\quad |b-c|_{\mathbb{Q}_5}$

$\left|\frac{3}{20} - \frac{35}{9}\right|_5 = \left|\frac{27 - 700}{20 \cdot 9}\right|_5 = 5.$

$$\left|\frac{17}{5}-\frac{3}{20}\right|_5 = \left|\frac{1}{5}\left(17-\frac{3}{4}\right)\right|_5 = 1.$$

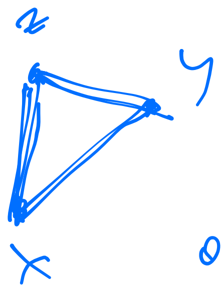
$$\frac{\cancel{68-3}\ 65}{4}.$$

$$\left|\frac{17}{5}-\frac{35}{9}\right|_5 = \left|\frac{153-175}{45}\right|_5 = 5.$$

Lemma: Every triangle in $\mathbb{Q}_p$ is isosceles.

Lemma: Super triangle inequality: $|x+y|_p \le \max(|x|_p, |y|_p)$.

ultrametric $d(x,y) \le \cancel{d(x,z)+d(z,y)}$.



$$\max\big(d(x,z),\, d(z,y)\big).$$

Sketch: $x = \overset{\neq 0}{a_{-N}}\, p^{-N} + \cdots$ $\qquad N > m.$

$y = \overset{\neq 0}{a_{-m}}\, p^{-m} + \cdots$

Open balls in $\mathbb{Q}_p$: $B(\overset{\in\mathbb{Q}_p}{x}, r)$

$= \{ y \in \mathbb{Q}_p : d(x,y) < r \}$. open

topology on $\mathbb{Q}_p$:= induced by $\mathcal{B}$ as a basis.

Ex: $B\left(x_0 = 2\cdot 5^{-2} + 1\cdot 5^{-1} + 3\cdot 5^{0} + 2\cdot 5^{1} + 3\cdot 5^{2} + \dots, \frac{1}{10}\right)$ in $\mathbb{Q}_5$.

$\overset{?}{=} \{ y = \underbrace{2\cdot 5^{-2} + 1\cdot 5^{-1} + 3\cdot 5^{0} + 2\cdot 5^{1}}_{\text{restricted}} + \underbrace{a_2\cdot 5^{2} + a_3\cdot 5^{3} + \dots}_{\text{free}} \}$

$d(x_0, 2\cdot 5^{-2} + 2\cdot 5^{-1} + \dots) = 5$.

$\{0,1\}^{\mathbb{N}} = \{ \underbrace{H H T T T H}\underbrace{T, \dots} \}$

cylinder sets

In $\mathbb{Q}_p$, balls are cylinder set.

If $y \in B(x,r)$ then $B(y,r) = B(x,r)$.

i.e. any interior pt of a ball is a center.

Is $\mathbb{Q}_p$ Hausdorff? ($=T_2$) Is it true that $\forall\, x \neq y \in \mathbb{Q}_p$, $\exists\, U, V$ open s.t. $U \cap V = \emptyset$ & $x \in U$, $y \in V$.

free U

$$x = 2 \cdot 5^{-3} + 1 \cdot 5^{-2} + 0 \cdot 5^{-1} + \boxed{1 \cdot 5^{0} + \cdots}$$

$$y = 2 \cdot 5^{-3} + 1 \cdot 5^{-2} + 1 \cdot 5^{-1} + \boxed{1 \cdot 5^{0} + \cdots}$$

free V

Is $\mathbb{Q}_p$ compact? No. $\mathbb{Q}_p \subset \bigcup_j \mathcal{O}_j$

$$\mathcal{O}_0 = \{\underbrace{a_0 \cdot p^0 + a_1 p^1 + \cdots}\},$$

$$\mathcal{O}_1 = \{a_{-1} \cdot p^{-1} + \mathcal{O}_0\}$$

$\mathcal{O}_j \subset \mathcal{O}_{j+1}$

$$\mathcal{O}_2 = \{a_{-2} \cdot p^{-2} + \mathcal{O}_1\},$$

If there were a finite subcover it would be a single $\mathcal{O}_N$. $\times$.

In $\mathbb{R}$, $\mathbb{Z} = \{ a_N \cdot 10^N + a_{N-1} \cdot 10^{N-1} + \dots + a_0 \cdot 10^0 +$ nothing after ".". $\cancel{a_{-1} \cdot 10^{-1} + \dots}$ $\}$

In $\mathbb{Q}_p$, $\mathbb{Z}_p = \{ \cancel{a_{-N} \cdot p^{-N} + a_{-N+1} \cdot p^{-N+1} + \dots + a_{-1} \cdot p^{-1}} + a_0 \cdot p^0 + a_1 \cdot p^1 + a_2 \cdot p^2 + \dots \}$

Is $\mathbb{Z}_p$ a ring? Yes.

$$\mathbb{Z}_p = B(0, 1+\varepsilon) = \bar{B}(0,1) = \{ x \in \mathbb{Q}_p, \ |x|_p \le 1 \}$$

$\mathbb{Z}_p = \varprojlim \mathbb{Z}/p^m\mathbb{Z}$.

$$\begin{tikzcd} & \mathbb{Z}/5^2\mathbb{Z} \arrow[r, "\bmod 5"] & \mathbb{Z}/5\mathbb{Z} \\ & \mathbb{Z}/5^3\mathbb{Z} \arrow[u, "\bmod 5^2"] \arrow[ur, "\bmod 5"'] & \end{tikzcd}$$

$x \longleftrightarrow (a_0, \ a_0 + a_1 p, \ a_0 + a_1 p + a_2 p^2, \dots)$

Is $\mathbb{Z}_p$ compact? Yes.

$\Updownarrow$ Sequential compactness i.e. every $x = \{x_n\} \subset \mathbb{Z}_p$

has convergent subseq.

Some a_0 occurs infinitely often, fix a_0, Let $X^0 \subset X$ be $\{x_n \in X, a_0(x_n) = a_0\}$.

$X^2 \subset X^1 \subset X^0$

X^1 ← same a_0 & a_1. Let $y_n = x_n^{(n)} \to_m \mathbb{Z}_p$,

X^2 ← same a_0 & a_1 & a_2.

Is $\mathbb{Q}_p$ locally compact, i.e. $\forall x \in \mathbb{Q}_p \ \exists x \in U$ open & K cpt $\supset U$? Yes.

Given $x = 3 \cdot 7^{-3} + 2 \cdot 7^{-2} + \underbrace{4 \cdot 7^{-1} + 5 \cdot 7^{0} + \cdots}_{\text{free coeff. } U}$

$U = 3 \cdot 7^{-3} + 2 \cdot 7^{-2} + \frac{1}{7} \cdot \mathbb{Z}_p = K$.

What is $\mathbb{Z}_p^\times = \{x \in \mathbb{Z}_p : \exists y \in \mathbb{Z}_p \text{ s.t. } x \cdot y = 1\}$

$\boxed{\begin{array}{l} |x \cdot y|_p = |1|_p = 1 \\ \Downarrow \\ |x|_p \cdot |y|_p = 1 \Rightarrow |x|_p = 1 \\ \le 1 \quad \le 1 \end{array}}$

$\mathbb{Z}_p \setminus \{0\}$, $= \{a_0 \cdot p^0 + a_1 \cdot p^1 + \cdots\}$, $a_0 \neq 0$. $= \{x \in \mathbb{Z}_p : |x|_p = 1\}$.

$x = 0 \cdot 3^0 + 1 \cdot 3^1 + 2 \cdot 3^2 + 1 \cdot 3^3 + \cdots \in \mathbb{Z}_p$.

$\notin \mathbb{Z}_p^\times$

$x = 2\cdot 3^0 + 1\cdot 3^1 + 2\cdot 3^2 + 1\cdot 3^3 + \cdots \in \mathbb{Z}_3$.

Find $x^{-1} = y = \overset{2}{\cancel{a_0}}\cdot 3^0 + \overset{0}{\cancel{a_1}}\cdot 3^1 + \overset{2}{\underline{\underline{a_2}}}\cdot 3^2 + \cdots$

$1 = x\cdot y \bmod 3 = 2\cdot 3^0 \cdot a_0\cdot 3^0 = 2a_0$, $a_0 = 2$

$1 = x\cdot y \bmod 9 = (2\cdot 1 + 1\cdot 3)(2+3a_1)$

$\boxed{\begin{array}{r} 147 \\ \times\ 295 \\ \hline \end{array}}$

$1 = 1 + 6a_1$ $\quad 6a_1 \equiv 0\ (9)$

$2a_1 \equiv 0\ (3)$.

$1 = x\cdot y \bmod 27 = (\underline{2} + 3 + 2\cdot 9)(2 + 0 + \underline{a_2\cdot 9})$

$= -4(2 + 9a_2) = -8 - 9a_2$

$= 19 + 18a_2$. $\Rightarrow$ $0 = 18 + 18a_2\ (27)$

$\Rightarrow a_2 = 2$. $\quad\Rightarrow 0 = 2 + 2a_2\ (3)$.

-1 in $\mathbb{Z}_3$? $\quad x+1=0$ in $\mathbb{Z}_3$.

$$x = 2 + \underbrace{2\cdot 3 + 2\cdot 3^2 + 2\cdot 3^3 + \ldots}_{3x} \quad \underbrace{(2+9\cdot 3+1 \equiv 0 (9)}$$

$$x = 2 + 3x \Rightarrow x = -1.$$

$$x = 2\big(\underbrace{1+3+3^2+\ldots}_{\frac{1}{1-3}}\big) = -1.$$