Last time: Modular forms & Applications.

Def: If $f : \mathbb{H} = \{ z = x+iy \mid y>0 \} \to \mathbb{C}$,

hol'c & $\exists\, k$, s.t. ⊛ $\boxed{f\left(\frac{az+b}{cz+d}\right) = (cz+d)^k f(z)}$

$\forall\, z \in \mathbb{H}$, $\forall \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL_2(\mathbb{Z})$. &

then f is a modular form of weight k

for $\Gamma = SL_2(\mathbb{Z})$.

$\Rightarrow f(z) = a_0 + a_1 q + a_2 q^2 + a_3 q^3 + \dots \quad q = e^{2\pi i z}$

⊛ for $\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$, $\underbrace{f\left(\frac{1\cdot z+1}{0\cdot z+1}\right)}_{f(z+1)} = (0\cdot z+1)^k f(z) = f(z)$.

i.e. $f(x+iy) = f(\underbrace{x+1}+iy) = \sum_{n\in\mathbb{Z}} \underbrace{a_n(y)}\, \underbrace{e^{2\pi i n x}}$.

$\Leftrightarrow$ Cauchy-Riemann $\frac{\partial f}{\partial \bar{z}} = 0$, $\underbrace{\frac{\partial}{\partial \bar{z}} = \frac{1}{2}\left(\frac{\partial}{\partial x} - \frac{1}{i}\frac{\partial}{\partial y}\right)}$.

Exercise: $\Rightarrow$ $a_n(y) = a_n \cdot \underbrace{e^{-2\pi n y}}$.

$$f(z) = \sum_{n \in \mathbb{Z}} a_n \cdot e^{2\pi i z}.$$

Fund domain for $\underbrace{(P)SL_2(\mathbb{Z})} \circlearrowright \mathbb{H}$.

$$\begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix} \in \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} : ad - bc = 1 \right\} / \mathbb{Z}.$$

Action of $(P)SL_2(\mathbb{R}) \circlearrowright \mathbb{H}$, fractional linear transf

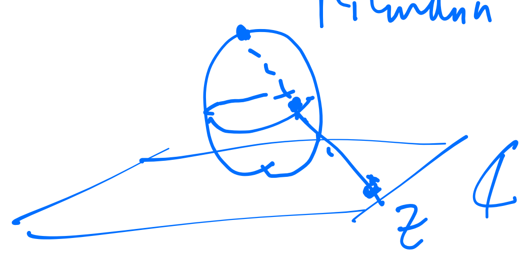
$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \circ z \mapsto \frac{az+b}{cz+d} \overset{??}{\in} \mathbb{H}.$$

Linear action on $\mathbb{P}^1\mathbb{C}$, $= \left\{ \underbrace{\begin{pmatrix} z \\ w \end{pmatrix}}_{\mathbb{C}^2 \setminus \{(0,0)\}} \right\} \Big/ \begin{pmatrix} z \\ w \end{pmatrix} \sim \begin{pmatrix} \lambda z \\ \lambda w \end{pmatrix}, \; \lambda \neq 0$, $\setminus (0,0)$.

$=$ If $w \neq 0$, $\rightarrow \begin{pmatrix} z/w \\ 1 \end{pmatrix} = \begin{pmatrix} \mathbb{C} \\ 1 \end{pmatrix}$

If $w = 0$, $\begin{pmatrix} 1 \\ 0 \end{pmatrix} = \infty$.

Riemann sphere



$$\begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} z \\ w \end{pmatrix} = \begin{pmatrix} az+bw \\ cz+dw \end{pmatrix}$$ usual. Linear action.

when $w=1$, $\begin{pmatrix} az+b \\ \underbrace{cz+d}_{\text{if } \neq 0} \end{pmatrix} \sim \begin{pmatrix} \frac{az+b}{cz+d} \\ 1 \end{pmatrix}$

(If $z \in \mathbb{H}$ & $c,d \in \mathbb{R}$ $\Rightarrow$ $cz+d \neq 0$).

$\boxed{\text{Exercise: } \operatorname{Im}\left(\frac{az+b}{cz+d}\right) = \frac{\operatorname{Im} z\,(ad-bc)}{|cz+d|^2} \geq 0}$

$\underline{SL_2(\mathbb{R}) \curvearrowright \mathbb{H}}$, $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \cdot z \mapsto \frac{az+b}{cz+d} \in \mathbb{H}$?

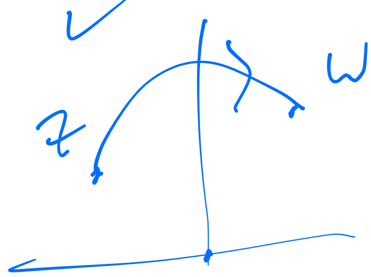
$\Downarrow$

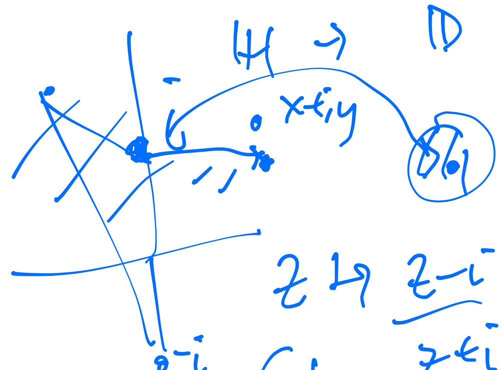
action? g_1, g_2, $g_1(g_2(z)) \overset{?}{=} (g_1 g_2)(z)$ ✓ Assoc of matrix mult!

✓

Is action transitive? $\forall z, w \in \mathbb{H}$, $\exists ?\, g \in SL_2(\mathbb{R})$

s.t. $g \cdot z = w$ ✓





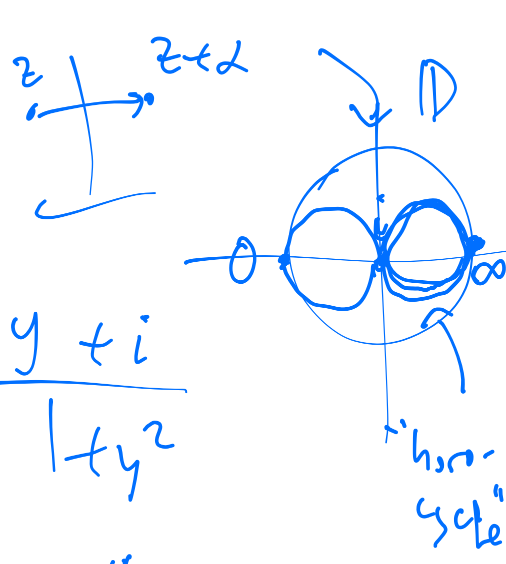
Given: $z = x+iy$

$z \mapsto \frac{z-i}{z+i}$ Cayley transform.

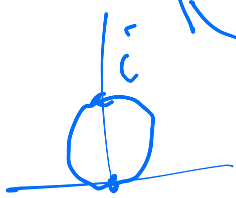
Need $g = (\quad)$ s.t. $g \cdot i = \frac{ai+b}{ci+d} = z$.

$$\begin{pmatrix} 1 & \alpha \\ 0 & 1 \end{pmatrix} \cdot z = \frac{1 \cdot z + \alpha}{0 \cdot z + 1} = z + \alpha.$$

$$g = \begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix} \circ i = i + x$$



$$\begin{pmatrix} 1 & 0 \\ y & 1 \end{pmatrix} \circ i = \frac{(i+0)}{(y\cdot i+1)}\frac{(1-yi)}{(1-yi)} = \frac{y+i}{1+y^2}$$



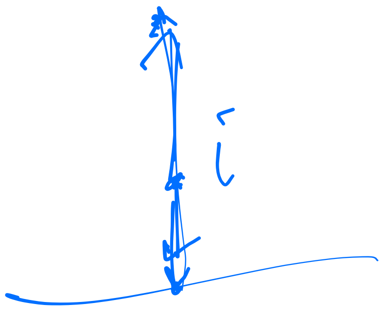
"parabolic" "unipotent"

"nilpotent" = powers become 0.

exp(nilpotent) = unipotent. all e-values 1.

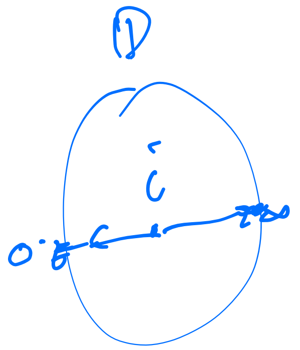
$\boxed{\text{Does transposition give action } y\text{-axis symmetry on } \mathbb{D}?}$ $\boxed{\text{No}}$.

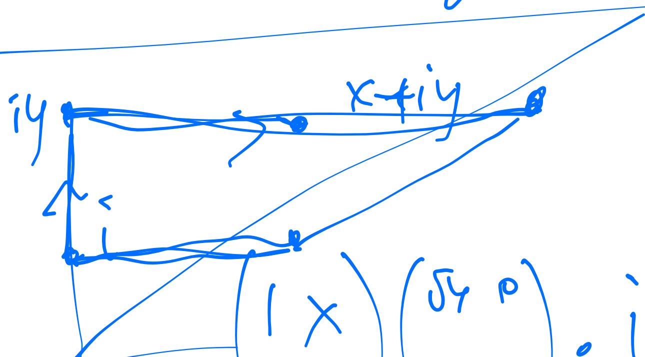
diagonalizable $\underbrace{\begin{pmatrix} \sqrt{y} & 0 \\ 0 & 1/\sqrt{y} \end{pmatrix}}_{y>0.} \circ i = \frac{\sqrt{y}\cdot i + 0}{0\cdot i + 1/\sqrt{y}} = y\cdot i$



$$\begin{pmatrix} y & 0 \\ 0 & 1 \end{pmatrix} \circ i = \frac{y\cdot i + 0}{0+1} = i\,y$$

GL_2^+.





$$\begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix}\begin{pmatrix} \sqrt{y} & 0 \\ 0 & 1/\sqrt{y} \end{pmatrix} \circ i = x + iy.$$

classification of motions

- unipotent $\sim \begin{pmatrix} 1 & * \\ 0 & 1 \end{pmatrix}$ (1 double real).
- hyperbolic $\sim \begin{pmatrix} y & 0 \\ 0 & 1/y \end{pmatrix}$ (2 real evals).
- elliptic $\sim \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix}$ (complex evals)

$\begin{pmatrix} a & b \\ c & d \end{pmatrix}$ eigenvalues $\lambda, \frac{1}{\lambda}$.

quad form:

$\underbrace{x^2+y^2}_{\shortparallel} = Q(x,y).$

$Q = (x\ y)\begin{pmatrix}1 & \\ & 1\end{pmatrix}\begin{pmatrix}x\\ y\end{pmatrix}$

$SO(p,q)$

$\underbrace{SO(2)} = \{k_\theta\}$

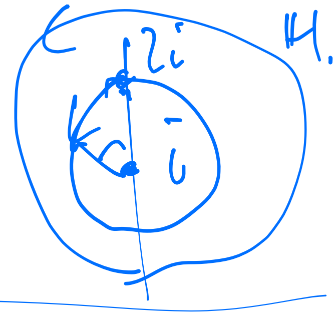
$\underbrace{\underbrace{\begin{pmatrix}\sqrt{y} & 0\\ 0 & 1/\sqrt{y}\end{pmatrix}\underbrace{\begin{pmatrix}1 & x\\ 0 & 1\end{pmatrix}\cdot i}}} = y\cdot i + \underline{y\cdot x}.$

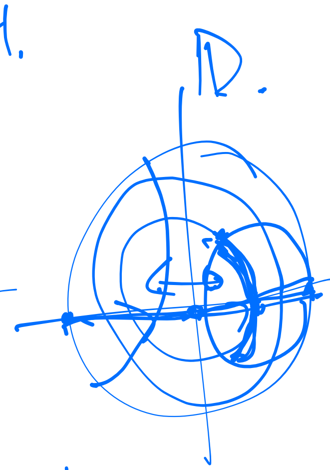
$\begin{pmatrix}\sqrt{y} & \sqrt{y}\cdot x\\ 0 & 1/\sqrt{y}\end{pmatrix}\cdot i = \dfrac{\sqrt{y}\cdot i + \sqrt{y}\cdot x}{1/\sqrt{y}}$

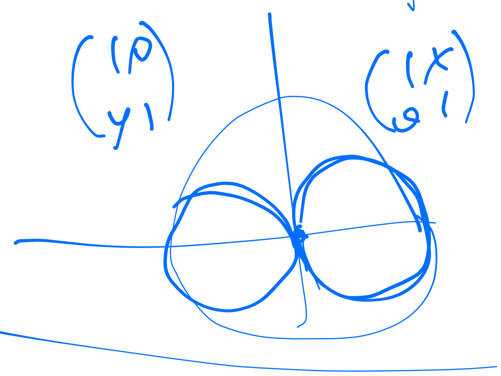
elliptic

$k_\theta = \begin{pmatrix}\cos\theta & \sin\theta\\ -\sin\theta & \cos\theta\end{pmatrix}\cdot i = \dfrac{(c\cdot i + s)}{(-s\cdot i + c)}$

$= \dfrac{i\, e^{-i\theta}}{e^{-i\theta}} = i.$





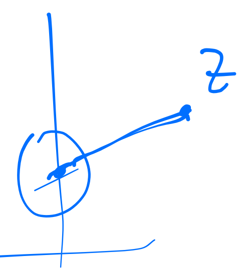


In general, if $G \circlearrowright X$

$G\cdot x_0 = X.$

is transitive,

$X \cong G/\underbrace{\mathrm{Stab}}_{x_0}.$



$\forall x, y.$

$(ax+by)^2 + (cx+dy)^2 = x^2+y^2.$

Q general quad form.

$\mathbb{H} \cong SL_2(\mathbb{R})/$

$x+iy \mapsto \begin{pmatrix}1 & x\\ 0 & 1\end{pmatrix}\begin{pmatrix}\sqrt{y} & 0\\ 0 & 1/\sqrt{y}\end{pmatrix}K.\ \underbrace{SO(2)},$

$O_Q(\mathbb{R}) = \{M : M^t Q M = Q\}.$

$gK_i \longleftarrow gK$

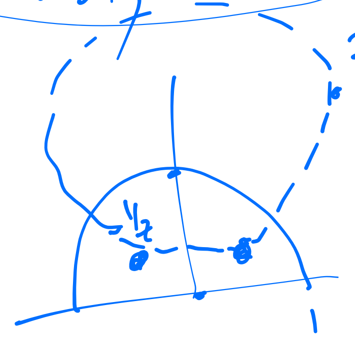
$\underline{\underline{x^t Q x}} = x^t \underbrace{M^t Q M x}$

$$\begin{array}{l} \begin{pmatrix} a & c \\ b & d \end{pmatrix} I \begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}. \\ a^2+c^2=1,\ b^2+d^2=1,\ ab+cd=0. \end{array}$$

$$SL_2(\mathbb{Z}) \ni \begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix}$$

$$SL_2(\mathbb{Z}) = \Big\langle \underset{T}{\overset{N=\begin{pmatrix}1 & \mathbb{R}\\ 0 & 1\end{pmatrix}}{\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}}},\ \underset{S}{\overset{K}{\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}}} \Big\rangle$$

$$\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \cdot \underset{\mathbb{H}}{z} = \frac{0 \cdot z + 1}{-z+0} = \frac{-1}{z}.$$



$$\frac{z}{|z|^2} = \frac{1}{\bar z}$$

$$\frac{-1}{\bar z}$$

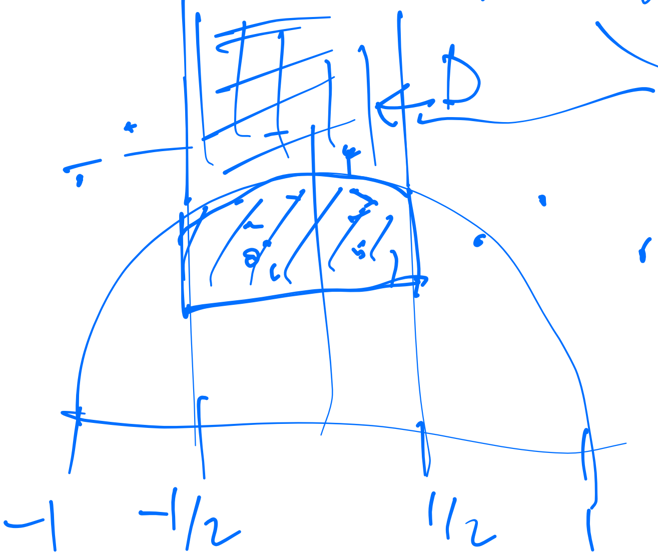
$$\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} = \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}.$$

Claim: If $|z|<1$, then

$$\operatorname{Im}\left(\frac{-1}{z}\right) > \operatorname{Im} z.$$

$$\boxed{\begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix} \cdot z = \frac{-z+0}{0-1} = z.}$$

Fund dom for $SL_2(\mathbb{Z}) \curvearrowright \mathbb{H}$:



$$\begin{pmatrix} 1 & \pm 1 \\ 0 & 1 \end{pmatrix}$$

z: shift z until $|\mathrm{Re}| \le 1/2$.

• If $|z| < 1$, apply S, else halt.

Action of $SL_2(\mathbb{Z})$ is discrete (is continuous). No limit points in K, operation must halt in finite time.