Fun with Modular Forms:

q-series $a_0 + a_1 q + a_2 q^2 + \cdots$ have "humongous symmetry"

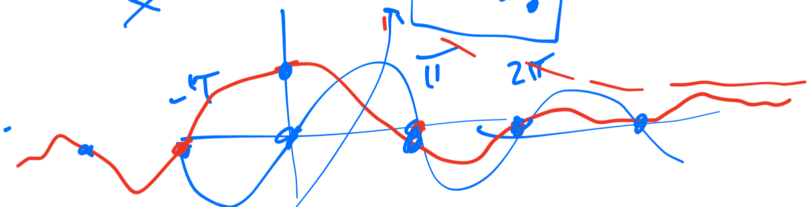
Prehistory: 1730s Euler solves Basel Problem: $\boxed{1+\frac{1}{8}+\frac{1}{27}+\frac{1}{64}+\cdots = ??}$

Bernoulli's $\sum \frac{1}{n^2} = \frac{1}{1}+\frac{1}{4}+\frac{1}{9}+\frac{1}{16}+\cdots = ? = \boxed{\frac{\pi^2}{6}}$.

$$\left(1-\frac{x^2}{\pi^2}\right)\left(1-\frac{x^2}{4\pi^2}\right)\left(1-\frac{x^2}{9\pi^2}\right)\cdots$$

$$\left(1-\frac{x}{\pi}\right)\left(1+\frac{x}{\pi}\right)\left(1-\frac{x}{2\pi}\right)\left(1+\frac{x}{2\pi}\right)\cdots = \frac{\sin x}{x} = 1 \boxed{-\frac{x^2}{3!}} + \frac{x^4}{5!} - \frac{x^6}{7!} - \cdots$$

$$-\frac{1}{\pi^2} - \frac{1}{4\pi^2} - \frac{1}{9\pi^2} - \frac{1}{16\pi^2} - \cdots = \boxed{-\frac{1}{6}}.$$



Compare terms $\boxed{x^2}$.

Knows: if p poly of deg d, roots $a_1,\dots,a_d \in \mathbb{R}$, & $p(0)=1 \Rightarrow p(x) = \left(1-\frac{x}{a_1}\right)\cdots\left(1-\frac{x}{a_d}\right)$

infinite product vs. infinite series ⇝ stuff.

$$\sum_{n\ge 1} \frac{1}{n^s} = \prod_p \left(1-\frac{1}{p^s}\right)^{-1} = \prod_p \left(1+\frac{1}{p^s}+\frac{1}{p^{2s}}+\frac{1}{p^{3s}}+\cdots\right)$$

$$\sum \frac{1}{p} = \frac{1}{2}+\frac{1}{3}+\frac{1}{5}+\frac{1}{7}+\frac{1}{11}+\frac{1}{13}+\cdots = \infty.$$

$$\frac{1}{1-\lambda} = 1+\lambda+\lambda^2+\lambda^3+\cdots$$

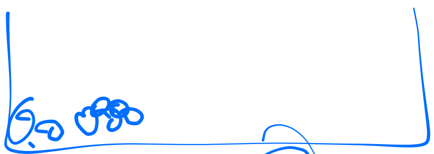
Partition function $p(5)=7$. $5 = 4+1 = 3+2 = 3+1+1 = 2+2+1 = 2+1+1+1 = 1+1+1+1+1$

Euler: $1 + p(1)q + p(2)q^2 + p(3)q^3 + \cdots = \sum p(n)q^n$.

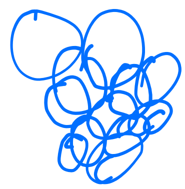
$$\frac{1}{(1-q)(1-q^2)(1-q^3)(1-q^4)(1-q^5)\cdots} = (1+q+q^2+q^3+\cdots)(1+q^2+q^4+q^6+\cdots)\cdot(1+q^3+q^6+q^9+\cdots)(1+q^4+\cdots)(1+q^5+\cdots)$$

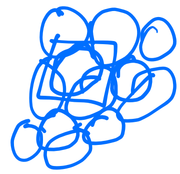
Coeff of q^5? $= \sum p(n)q^n$.

① Kepler 1611: equal sphere packings, densest possible configuration?



Both pack 74% of space. Can one do better?..





$\frac{\pi}{3\sqrt{2}}$

1998 Hales proof Kepler. (Fejes Toth 40/50s)

massive computation. Annals: 99% sure.

2003 initiates FlysPecK Formal Proof Kepler.

2014 Complete formal pf, lots of collaborators.

3-D. 2-D Thue 1890s  optimal Fejes Toth 40.

4-D ??? 5-D ?? ...

8-D, E_8 lattice $(n_1, \dots, n_8) \in \mathbb{R}^8$ $\left\{ \begin{array}{l} \forall n_i \in \mathbb{Z} \\ \forall n_i \in \mathbb{Z} + \frac{1}{2} \end{array} \right.$ $\sum n_i$ even.

distance to nearest pt is $\sqrt{2}$.

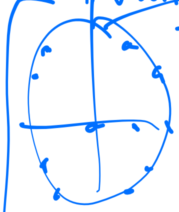
2016 Viazovska: E_8 lattice is densest. 2022 F.M.

2016 " + ... + S.D. Miller (24-D) Leech lattice

Hints: $\#\{\text{pts at distance } n \text{ in } E_8 \text{ ; } \sum n_i^2 = 2n\}$ Fact:

$$= 240 \cdot \sum_{d \mid n} d^3$$

Fermat: $\#\{(x,y) \in \mathbb{Z}^2 : x^2+y^2 = n\}$



$$\boxed{E_4(z) := 1 + \sum_{n \geq 1} \underbrace{240 \cdot \sigma_3(n)}\, q^n .}$$

$E(z+1)$

$\sigma_3(n)$.

is a mod form.

$E\left(-\frac{1}{z}\right)$ $=$ $z^4 \cdot E(z)$

$|q| < 1$, $q \in \mathbb{D}$

, $q = e^{2\pi i z}$, $|q| = e^{-2\pi y} < 1$,

"nonobvious symmetry":

$z = x+iy \in \mathbb{H}$

$y > 0$. $z \mapsto \frac{-1}{z}$.

$-1/z$

$-\frac{z}{|z|^2}$ $\frac{z}{|z|^2}$

$z \to \frac{z}{|z|^2} = \frac{1}{\bar{z}}$

$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL_2(\mathbb{Z})$

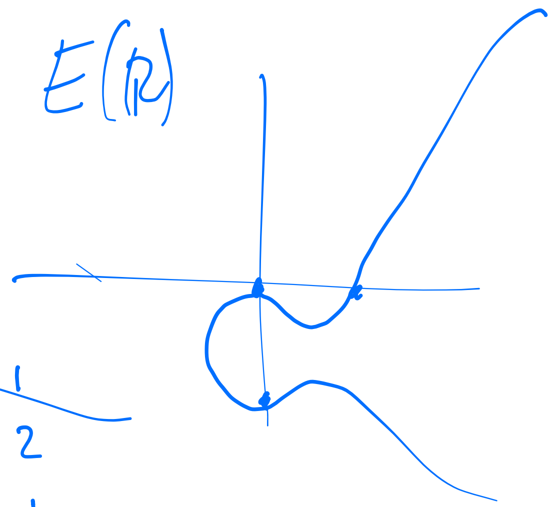
$E\left(\frac{az+b}{cz+d}\right) = (cz+d)^4 E(z)$

$$\vartheta(z) = \sum_{n \in \mathbb{Z}} e^{2\pi i n^2 z} = \sum_n q^{n^2} .$$

$z \in \mathbb{H}$

$$\vartheta(z)^2 = \sum_{n,m} q^{n^2+m^2} = \sum_{k \in \mathbb{Z}} \#\{k = n^2+m^2\} \cdot q^k$$

② $E: \underbrace{y^2+y=x^3-x^2}$ $E(\mathbb{R})$



$\#\{(x,y)\in \mathbb{Z}/p\mathbb{Z} : y^2+y \equiv x^3-x^2 \pmod{p}\}$

$= p + a(p)$ $p=3$

			x	0	1	2
			x^2	0	1	1
			x^3	0	1	2
			$\to x^3-x^2$	0	0	1
y^2+y	y^2	y				
0	0	0		$\otimes$	$\otimes$	
2	1	1				
0	1	2		$\otimes$	$\otimes$	

$4 = 3 + a(3)$,

$a(3)=1$.

$L_E(s) = \prod_{p\nmid\Delta}\left(1-\frac{a(p)}{p^s}+\frac{p}{p^{2s}}\right)^{-1}$

$= \sum \frac{\boxed{a(n)}}{n^s}$.

$f_E(z)$ "=" $\displaystyle\sum_{n\geq 1} a(n)q^n$. $\leftarrow$ is modular form!

50s Taniyama-Shimura-Weil "Modularity" (Langlands),

Conj: Every elliptic curve is modular.

80s Frey: If $a^p+b^p=c^p$, $p\geq 3$, $(abc)\neq 0$, $a,b,c\in\mathbb{Z}$

ie if Fermat's Last Thm false, then:

$y^2 = x(x-a^p)(x+b^p)$ ~~might~~ will not be modular!!!

Serre + Ribet

A. Wiles proved enough Modularity $\Rightarrow$ FLT. (Wiles-Taylor)

③ $\underbrace{x^2+y^2=1}$ parametrization? $\underbrace{x=\cos\theta}$, $\underbrace{y=\sin\theta}$,

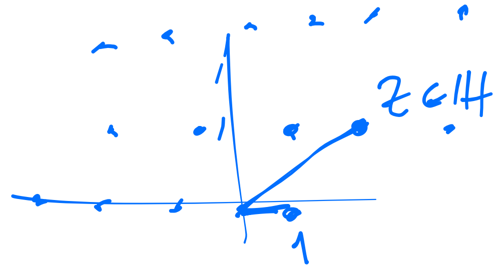
$\theta \in \mathbb{R}/2\pi\mathbb{Z}$ ← lattice.

$\underbrace{y^2=x^3+Ax+B}$ parametrization?

Weierstrass $x=\wp_\Lambda(w)$, $y=\wp'_\Lambda(w)$. $\Lambda \overset{=\langle 1,z\rangle}{}$ lattice $\subset \mathbb{C}$.

$$\wp'(w)^2 = \wp(w)^3+\underbrace{E_4(z)}\cdot\wp(w)+\underbrace{E_6(z)}$$

$z \in \mathbb{H}$



$w \in \mathbb{C}/\Lambda$.

"j-invariant"

$$j(z)=\frac{E(z)^3}{\Delta(z)}$$

$$\boxed{\Delta(z)=q\prod_{n\ge 1}(1-q^n)^{24}}$$

Deligne Fm

✓ Mordell $|\tau(p)|\le 2p^{11/2}$.

Ramanujan 1917 = $\tau(m\cdot n)=\tau(m)\cdot\tau(n)$, $(m,n)=1$

$q-24q^2+252q^3-1472q^4+4830q^5-6048q^6\cdots$

$=\sum \tau(n)q^n$.

$$=\frac{\left(1+\sum_{n\ge 1} 240\cdot\sum_{d\mid n} d^3\cdot q^n\right)}{q(1-q)^{24}(1-q^2)^{24}\cdots}$$

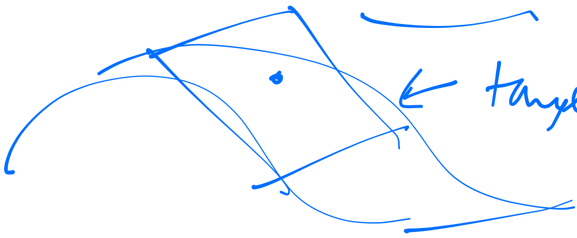
$$j\left(\frac{1+\sqrt{163}i}{2}\right)$$

$$\overset{\shortparallel}{-640320^3} = q^{-1}+744+\underline{196884}\,q+\underbrace{21{,}493{,}760}\,q^2+\cdots$$

Aside: 70's finite simple groups $PSL_2(\mathbb{Z}/17\mathbb{Z})$.

- finite list of infinite family.
- sporadic groups finite (26).

Largest: Monster 8×10^{53} elements.



← tangent space to manifold (linearization)

(irreducible) reps of Group ('').

dim

Irred rep of M: 1, 196883, 21296876, ---

Conway Moonshine $\xrightarrow{80}$ vertex operator algebras .. Jim Lepowsky.

Richard Borcherds proved, 1998 F.M.

Ramanujan: $j\left(\frac{1+\sqrt{163}\,i}{2}\right) = \underbrace{-640320^3}_{} \in \mathbb{Z}.$

$$\underbrace{q^{-1}} + \underline{744} + \underline{\qquad} q + \cdots$$

$$q = e^{2\pi i z} = e^{2\pi i\left(\frac{1}{2} + \frac{i\sqrt{163}}{2}\right)} = -e^{-\pi\sqrt{163}}$$

$$-e^{\pi\sqrt{163}} + 744 + \underline{\qquad}\, e^{-\pi\sqrt{163}} + \cdots$$

$$\Rightarrow e^{\pi\sqrt{163}} = 640320^3 + 744 + O\left(10^{-12}\right).$$