Last time:

Want to understand decomposition $L^2(\Gamma\backslash G, \mu)$, $G = GL(1, \mathbb{A})$, $\Gamma = GL(1, \mathbb{Q})$

into irreducibles. G = abelian, 1-dim'l

irreps $W = \mathbb{C}\cdot\phi$ ← automorphic representation. $\phi : \Gamma\backslash G \to \mathbb{C}$ ie.

$\forall g \in G = \mathbb{A}^\times$, $\forall \gamma \in \Gamma = \mathbb{Q}^\times$, $\phi(\gamma g) = \phi(g) = \phi(\Gamma g)$.

& $(\pi(g).\phi)(h) = \boxed{\phi(hg)} = \omega(g)\cdot\phi(h)$.

π ← right regular rep. → unitary $\langle \pi(g)\phi_1, \pi(g)\phi_2\rangle = \langle \phi_1, \phi_2\rangle$

Not true in general!! $GL(2, \mathbb{A}_{\mathbb{Q}})$

Irreps will be ∞ dim'l.

ω "Hecke character": $\mathbb{Q}^\times\backslash\mathbb{A}^\times \to S^1 \subset \mathbb{C}^\times$

$$\phi(g) = \phi(\gamma g) = \pi(g).\phi(\gamma) = \underbrace{\omega(g)\cdot\phi(e)}_{c}$$

ω "global" character $\mathbb{A}^\times$, $\omega(g_1 g_2) = \omega(g_1)\cdot\omega(g_2)$

Let $\omega_p(g_p) = \omega(1, 1, .., 1, g_p, 1, ..)$.

$g = (g_\infty, g_2, g_3, \dots)$ a.a. $g_p \in \mathbb{Z}_p^\times$

char on $\omega_p : \mathbb{Q}_p^\times \to S^1 \subset \mathbb{C}^\times$.

$$\omega(g) = \prod_{p\le\infty} \omega_p(g_p).$$

$$(g_\infty, 1, 1, \dots)\cdot(1, g_2, 1, 1\dots)\cdot(1, 1, g_3, 1\dots)\cdots = (g_\infty, g_2, g_3, \dots).$$

ω cont on $\underbrace{\mathbb{A}_f^\times}_{\text{"finite ideles"}} = \underbrace{\prod_{p<\infty}' \mathbb{Q}_p^\times}_{\text{totally disconnected}}$, $\mathbb{A}^\times = \mathbb{R}^\times \times \mathbb{A}_f^\times$.

$\operatorname{Ker}\omega \supset \prod_{p\in S} U_p \times \prod_{p\notin S} \mathbb{Z}_p^\times$

i.e. a.a. p, $\omega_p|_{\mathbb{Z}_p^\times} \equiv 1$

Def: Local char ω_p is <u>unramified</u> if $\forall u\in\mathbb{Z}_p^\times$, $\omega_p(u) = 1$.

<u>If ω_p is unramified:</u> $g_p \in \mathbb{Q}_p^\times$, $\Rightarrow g_p \in p^m\mathbb{Z}_p^\times$.

$$\omega_p(g_p) = \omega_p(p^m\cdot u)$$

$$= \omega_p(p)^m \cdot \cancel{\omega_p(u)}^{1}.$$

$\underset{\neq 0}{a_{-N}}\, p^{-N} + \dots + a_{-1}p^{-1} + a_0 + a_1 p + a_2 p^2 + \dots$

Then ω_p completely def'd by $\omega_p(p)$.

If ω_p is ramified, $\exists\, u\in\mathbb{Z}_p^\times$ s.t. $\omega_p(u)\neq 1$.

Then $\operatorname{Ker}\omega_p \supseteq 1 + p^{m_p}\mathbb{Z}_p$ with minimal $m \geq 1$.

$1 + \underbrace{a_1 p + a_2 p^2 + a_3 p^3 + \dots + a_m p^m}$

$\omega_p(1) = 1.$

We say that p^{m_p} (local) <u>conductor</u> of ω_p.

<u>Obs:</u> $1 + p^m\mathbb{Z}_p$ mult subgp of $\mathbb{Z}_p^\times$.

Look at $\mathbb{Z}_p^\times / (1 + p^m\mathbb{Z}_p) \cong \{ \underset{\neq 0}{a_0} + a_1 p + \dots + a_{m-1}p^{m-1} \} \cong (\mathbb{Z}/p^m\mathbb{Z})^\times$

So $\omega_p\big|_{\mathbb{Z}_p^\times} = \chi_{p^{m_p}}$, i.e. $\omega_p(a_0 + a_1 p + \cdots) = \chi_{p^{m_p}}(\underbrace{a_0 + a_1 p + \cdots + a_{m-1} p^{m-1}}_{\in (\mathbb{Z}/p^m\mathbb{Z})^\times})$.

$\exists$ Dirichlet char $\chi_{p^{m_p}}$ s.t.

Put them all together, let $N = \prod_{p \text{ ramified}} p^{m_p}$, then

$$\chi_{(\mathrm{mod}\ N)} = \prod \chi_{p^{m_p}}$$

$$(\mathbb{Z}/N\mathbb{Z})^\times \cong \prod_p (\mathbb{Z}/p^{m_p}\mathbb{Z})^\times.$$

s.t. $\omega\big|_{\prod \mathbb{Z}_p^\times} = \chi$ ← primitive!

Summary: From primitive χ mod N, can construct Hecke char

$\omega : \mathbb{Q}^\times\backslash\mathbb{A}^\times \to \mathbb{C}^\times$, $\omega(g) = \prod \omega_p(g_p)$, $\boxed{\omega\big|_{\mathbb{Q}^\times} = 1.}$

- If ω_p unramified ($p \nmid N$), then $\omega_p(p^m u) = \chi(p)^m$
- If ω_p ramified with conductor p^m, then $\omega_p(p^m \cdot u) = \chi(u \bmod p^m)$
- For $p = \infty$, $\omega_\infty(g_\infty) = |g_\infty|^{it} \cdot \begin{cases} 1 & \leftarrow \chi(-1) = 1 \\ \mathrm{sign}(\underset{\in \mathbb{R}^\times}{g_\infty}) & \leftarrow \chi(-1) = -1. \end{cases}$

  Exercise!

Given such an aut rep $\phi = \omega = \chi$, attach $L(\phi, s) = L(\chi, s)$.

To make ϑ, took "test vector" $f \in S(\mathbb{A})$,

$$\vartheta_f(\underset{\in\mathbb{A}^\times}{t}) := \sum_{q \in \mathbb{Q}^\times} f(qt) \left(= \sum_{n \in \mathbb{Z}\setminus\{0\}} f\left(\tfrac{n}{N} t\right)\right) \text{ on } \mathbb{Q}^\times\backslash\mathbb{A}^\times$$

Then: $\widetilde{\vartheta}_f(t) = \int_{\mathbb{Q}^\times \backslash \mathbb{A}^\times} \vartheta_f(t) \cdot \underbrace{|t|^s_{\mathbb{A}}}\, d^\times t \rightsquigarrow_{(\underline{\underline{\mathrm{Re}\, s > 1}})} \widetilde{f}_\infty(s) \cdot \zeta(s)$ $\quad \prod_p (1-\frac{1}{p^s})^{-1}$

(If $f_p = \mathbb{1}_{\mathbb{Z}_p}$).

"twisted adelic Mellin transform"

Now: $\int_{\mathbb{Q}^\times \backslash \mathbb{A}^\times} \vartheta_f(t)\, \omega(t)\, |t|^s_{\mathbb{A}}\, d^\times t \overset{\text{unfold}}{=} \int_{\mathbb{A}^\times} f(t)\, \omega(t)\, |t|^s\, d^\times t.$

$$= \prod_p \int_{\mathbb{Q}_p^\times} f_p(t_p)\, \omega_p(t_p)\, |t_p|_p^s\, d^\times t_p.$$

Normalized $\int_{\mathbb{Z}_p^\times} 1\, d^\times t_p = 1.$

Look at p (a.a.) with $f_p = \mathbb{1}_{\mathbb{Z}_p}$, ω_p = unramified.

$$\int_{\mathbb{Q}_p^\times} f_p(t_p)\, \omega_p(t_p)\, |t_p|_p^s\, d^\times t_p = \int_{\mathbb{Z}_p \backslash \{0\}} = \sum_{n \geq 0} \int_{p^n \mathbb{Z}_p^\times} \overset{\chi(p)^n}{\cancel{\omega_p(t_p)}}\, p^{-ns}\, d^\times t_p$$

$$= \sum_{n \geq 0} \left(\frac{\chi(p)}{p^s}\right)^n \cdot \overset{1}{\cancel{\int_{p^n \mathbb{Z}_p^\times} d^\times t_p}}. = \frac{1}{1 - \frac{\chi(p)}{p^s}} = L_p(\chi, s).$$
