Last time: "twisted" Poisson summation

$$\sum_{n\in\mathbb{Z}} f(n)\,\overline{\chi(n)} \quad \text{Dirichlet character mod } N.$$

If $h : \mathbb{Z}/N\mathbb{Z} \to \mathbb{C}$, $h \in L^2(\mathbb{Z}/N\mathbb{Z})$,

$$\underset{\mathbb{Z}/N\mathbb{Z}}{\hat{h}}(a) = \frac{1}{\sqrt{N}} \sum_{m (N)} h(m)\, e^{2\pi i \frac{am}{N}} = \frac{1}{\sqrt{N}} \sum_{m=0}^{N-1} h(m)\, e^{2\pi i \frac{am}{N}} \qquad a \in \mathbb{C},$$

Exercise: $h(m) = \underbrace{\frac{1}{\sqrt{N}} \sum_{a=0}^{N-1} \hat{h}(a)\, e^{-2\pi i \frac{am}{N}}}$ extension of h to $\mathbb{C}$-val.

Fix $t>0$

$$\sum_{n\in\mathbb{Z}} f(nt)\, h(n) = \frac{1}{\sqrt{N}} \sum_{a=0}^{N-1} \hat{h}(a) \sum_{n\in\mathbb{Z}} f(nt)\, e^{-2\pi i \frac{an}{N}}.$$

P.S. $\rightarrow$

$$= \frac{1}{\sqrt{N}} \sum_{a=0}^{N-1} \hat{h}(a) \sum_{m\in\mathbb{Z}} \int_{\mathbb{R}} f(xt)\, e^{-2\pi i \frac{ax}{Nt}}\, e^{2\pi i \frac{mx}{t}} \frac{dx}{t},$$

$x \mapsto x/t$.

$$= \frac{1}{t\sqrt{N}} \sum_{a=0}^{N-1} \hat{h}(a-mN) \sum_{m\in\mathbb{Z}} \hat{f}\left(\frac{Nm-a}{tN}\right)$$

as $a \,(N)$, $m \in \mathbb{Z}$, $Nm-a$ run over $k \in \mathbb{Z}$

$$\boxed{= \frac{1}{t\sqrt{N}} \sum_{k\in\mathbb{Z}} \hat{f}\left(\frac{k}{tN}\right) \hat{h}(-k) = \sum_{n\in\mathbb{Z}} f(nt)\, h(n)}$$

twisted (by h), dilated (t, t) Poisson sum formula.

Assume: f, h even. Then let

Also assume $h(0) = \hat{h}(0) = 0$

$$\vartheta_{f,h}(t) := \sum_{n\geq 1} f(nt)\, h(n) = \frac{1}{t\sqrt{N}} \vartheta_{\hat{f},\hat{h}(-)}\left(\frac{1}{tN}\right) - \cancel{\frac{f(0)h(0)}{2}} + \cancel{\frac{\hat{f}(0)\hat{h}(0)}{2t\sqrt{N}}}.$$

As before: $$\tilde{\vartheta}_{f,h}(s) = \int_0^\infty \left(\sum_{n\geq 1} f(nt)\, h(n)\right) t^s \frac{dt}{t}$$ $\mathrm{Re}(s) \gg 1$.

$$= \sum_{n\geq 1} \frac{h(n)}{n^s} \cdot \int_0^\infty f(nt)\, t^s \frac{dt}{t} = L(h,s) \cdot \tilde{f}(s).$$

$t \mapsto t/n$

$$= \int_{\boxed{K}}^{\infty} + \int_0^K \left(\frac{1}{t\sqrt{N}} \vartheta_{\hat{f},\hat{h}(-)}\left(\frac{1}{tN}\right)\right) t^s \frac{dt}{t}$$

TBD $\longrightarrow \boxed{K}$

$$y = \frac{1}{t \cdot N}, \quad t = \frac{1}{yN}$$

$$= \overbrace{\int_K^{\infty} \vartheta_{f,h}(t)\, t^s \frac{dt}{t}} + \int_{\frac{1}{KN}}^{\infty} \frac{1}{\sqrt{N}} \vartheta_{\hat{f},\hat{h}(-)}(y)\, y^{1-s} \cdot N^{1-s} \frac{dy}{y}$$

$$\boxed{K = \frac{1}{\sqrt{N}}}$$

$$N^{s/2} \tilde{\vartheta}_{f,h}(s) = N^{s/2} L(h,s) \tilde{f}(s) =$$

$$= \int_{\frac{1}{\sqrt{N}}}^{\infty} \left[N^{s/2} \vartheta_{f,h}(t)\, t^s + N^{\frac{(1-s)}{2}} \vartheta_{\hat{f},\hat{h}(-)}(t)\, t^{1-s} \right] \frac{dt}{t}.$$

entire in s.

Symmetric under $\begin{pmatrix} f \\ h \\ s \end{pmatrix} \longleftrightarrow \begin{pmatrix} \hat{f} \\ \hat{h}(-) \\ 1-s \end{pmatrix}$.

Analytic Cont & FE

Note: Has nothing to do with A.

Completely multiplicative.

If $\underline{h(n\cdot m) = h(n)\,h(m)}$, then, $\underbrace{\qquad}_{\mathrm{Re}\, s>1}$

FTA. $L(h,s) = \sum_{n\geq 1} \frac{h(n)}{n^s} = \prod_p \left(1 - \frac{h(p)}{p^s}\right)^{-1}$.

When $h = \chi$, $L(\chi, s)$ is $\underline{\text{the}}$ Dirichlet L-function.

$\underline{Q}$: Can $L(\chi, s) = 0$ for $\underbrace{\mathrm{Re}\, s > 1}$?

$\underline{\text{No!}}$ $\underline{\text{Euler Product.}}$

When $h = \chi$, can say $\underline{\text{more}}$. Look at

$$\hat{h}(a) = \hat{\chi}(a) = \frac{1}{\sqrt{N}} \sum_{m(N)} \chi(m)\, e^{2\pi i \frac{ma}{N}},$$

If $(a,N)=1$, then $m \mapsto m\bar{a}$ is permutation.

$$= \frac{1}{\sqrt{N}} \sum_{m(N)} \underbrace{\chi(m\bar{a})}\, e^{2\pi i \frac{m}{N}} = \overline{\chi(a)}\, \frac{1}{\sqrt{N}} \sum_{m(N)} \chi(m)\, e^{2\pi i \frac{m}{N}}$$

$$= \overline{\chi}(a)\,\hat{\chi}(1)$$

$\tau(\chi) =$ "Gauss sum".

Why? $\longrightarrow$

<u>Gauss:</u> $\chi(m) = \left(\frac{m}{N}\right) = \begin{cases} 0 & (m,N)\neq 1 \\ 1 & m = x^2\ (N) \\ -1 & m \neq x^2 \end{cases}$ quad residue symbol.

N prime

$$t(\chi) = \sum_{m\,(N)} \underbrace{\left(1 + \chi(m)\right)}_{\begin{cases} 1 & m=0 \\ 2 & m = x^2 \\ 0 & \text{else.} \end{cases}} e^{\frac{2\pi i m}{N}} = \sum_{x\,(N)} \underbrace{e^{\frac{2\pi i x^2}{N}}}_{e^{-\pi x^2}.}$$

$$\hat{\chi}(a) = \overline{\chi}(a) \cdot \frac{\tau(\chi)}{\sqrt{N}}.$$

$\leftarrow$ Correct for all a if <u>χ primitive.</u>

$$N^{s/2} L(\chi, s)\, \tilde{f}(s)$$

$$= N^{\frac{1-s}{2}} L(\hat{\chi}(-), 1-s)\, \widetilde{\hat{f}}(1-s).$$

$\leftarrow$ in general

$$= \left(\frac{\tau(\chi)}{\sqrt{N}}\right) N^{\frac{1-s}{2}} L(\overline{\chi(-)}, 1-s)\, \widetilde{\hat{f}}(1-s)$$

$\leftarrow$ if primitive

"root number" "epsilon factor",

Exercise: $h : \mathbb{Z}/N\mathbb{Z} \to \mathbb{C}$, Parseval identity

$\wedge$ is unitary, i.e. $\langle h_1, h_2 \rangle = \langle \hat{h}_1, \hat{h}_2 \rangle$

$$\frac{1}{\sqrt{N}} \sum_{m (N)} h_1(m) \overline{h_2(m)} = \frac{1}{\sqrt{N}} \sum_{a (N)} \hat{h}_1(a) \overline{\hat{h}_2(a)}.$$

$h_1 = h_2 = \chi$, primitive

$$\frac{1}{\sqrt{N}} \phi(N) = \langle \chi, \chi \rangle = \langle \hat{\chi}, \hat{\chi} \rangle = \frac{1}{\sqrt{N}} \left| \frac{\tau(\chi)}{\sqrt{N}} \right|^2 \phi(N),$$

Euler totient $= |\mathbb{Z}/N^\times|$.

$$\Rightarrow \quad |\tau(\chi)| = \sqrt{N}.$$

$$N^{s/2} L(\chi, s)\, \gamma(s) = \frac{\tau(\chi)}{\sqrt{N}} N^{\frac{1-s}{2}} L(\overline{\chi}(-), 1-s)\, \tilde{\gamma}(1-s),$$

If $\chi(-m) = \chi(-1)\chi(m)$, χ is even / odd iff $\chi(-1) = 1$ / -1

$$N^{s/2} L(\chi,s)\underbrace{\tilde{f}(s)} = \frac{\chi(-1)\tau(\chi)}{\sqrt{N}} N^{\frac{1-s}{2}} L(\bar{\chi},1-s)\tilde{\hat{f}}(1-s)$$

h'f even, if χ even, $f(x) = e^{-\pi x^2}$,

if χ odd, $f(x) = x e^{-\pi x^2}$

Back to adeles / ideles,

$G = GL(1,\mathbb{A}) = \mathbb{A}^\times$

$\Gamma = GL(1,\mathbb{Q}) = \mathbb{Q}^\times$) discrete

$$\Gamma \backslash G = \mathbb{Q}^\times \backslash \mathbb{A}^\times \cong (0,\infty) \times \prod_p \mathbb{Z}_p^\times.$$

<u>Want</u>: decompose $G \overset{\text{right-reg}}{\circlearrowright} L^2(\Gamma\backslash G, d\mu^\times)$ rep. into

<u>irreducibles</u> (1-dim'l, since G = abel'ian).

Automorphic reps. $W = \mathbb{C}\underbrace{\phi}$.

$\phi : \mathbb{A}^\times \to \mathbb{C}$ s.t. $\phi(\gamma g) = \phi(g)$
$\forall g \in G = \mathbb{A}^\times,\ \gamma \in F = \mathbb{Q}^\times$,

W is G-invar

& $(\pi(g).\phi)(h) := \phi(h.g) = \omega(g).\phi(h)$,

for some continuous $\omega : \mathbb{A}^\times \to \mathbb{C}^\times$.

$\mathbb{R} \curvearrowright L^2(\mathbb{R}, dx) =: \mathcal{H} \circlearrowleft \Delta = \frac{\partial^2}{\partial x^2}$, = linear op on $\mathcal{H}$,

$\cup$

$f \in C_c^\infty(\mathbb{R}, dx)$

$$\underbrace{f(x)}_{} = \int_{\operatorname{Spec}\Delta} \overbrace{\underbrace{\hat{f}(\xi)}_{}\, \underbrace{e^{-2\pi i x \xi}}_{}}^{= \operatorname{Proj}_{e_\xi} f} d\xi,$$

$e_\xi \notin L^2(\mathbb{R})$,

When is $\underbrace{L - \lambda \cdot I}_{: \mathcal{H} \to \mathcal{H}}$ $\lambda \in \mathbb{C}$ an invertible

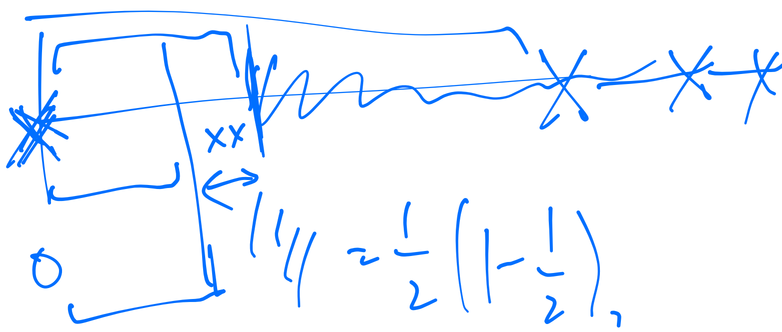
$$\operatorname{Spec} L = \{ \lambda \in \mathbb{C} \text{ s.t. } L - \lambda I \text{ not invertible} \}.$$

Not invertible: $\exists$ kernel, $v \in \mathcal{H}$ s.t. $(L - \lambda I)v = 0$, eigenvectors ("discrete spectrum").

OR not onto $\Rightarrow$ $(L - \lambda I)$ not invertible.

λ is "continuous spectrum", "Selberg 1/4".

$\Delta \circlearrowleft L^2\left(SL_2(\mathbb{Z}) \backslash \mathbb{H}\right)$, with $\Gamma(N)$ under $SL_2(\mathbb{Z})$



$$\frac{1}{4} = \frac{1}{2}\left(1 - \frac{1}{2}\right),$$

Kim-Sarnak!

$$= \left(\frac{1}{2} - \frac{7}{64}\right)\left(\frac{1}{2} + \frac{7}{64}\right).$$