Last time: Crash Course in Rep Th.

G = group (loc cpt top grp / finite)

μ = Haar measure, $L^2(G, \mu) =: V$

$= \{ f : G \to \mathbb{C} : \int_G |f|^2 d\mu < \infty \} / \sim .$

Working through decomposition of $\pi = \pi_{reg} = (G, V)$ unitary rep.

into irreducibles, where $(\pi(g).f)(k) = f(kg)$.   ($g \in G$, $f \in V$)

$(\pi(gh).f)(k) = \pi(g) f_h(k) = f(kgh)$.

$(\pi(gh).f)(k) = f(kgh)$, $f_h(k) = f(kh)$ $\pi : G \to GL(V)$

Ex. $G = SL_2(\mathbb{F}_2) \cong S_3 \cong D_6$

$= \langle a, b \mid a^3 = e = b^2, \; ba = a^2 b \rangle$.

$V = L^2(G) = \mathbb{C}^{|G|}$ "group basis" $\{ e, [a, a^2], [b, ab, a^2 b] \}$

ord 3 ord 2

Found: $e_1 = (1,1,1,1,1,1,1)$, $W_1 = \mathbb{C} e_1$ 1-dim irred.

Def: Given $\pi = (G, W)$, χ = character of $\pi : G \to \mathbb{C}$.

$\chi(g) := \mathrm{tr}(\pi(g))$. Obs: χ is constant on conjugacy classes.

$$\chi(hgh^{-1}) = \mathrm{tr}(\pi(h)^{-1} \pi(g) \pi(h)) = \chi(g)$$

Facts:

① $\chi_{reg} = \begin{cases} |G| & g = e \\ 0 & g \neq e \end{cases}$, ② for any (G,W), $\chi(e) = \dim W$.

Claim: Given irred $\pi = (G, W)$, mult{with which π occurs in decomp of $V = L^2(G)$} $= \dim W$.

pf: $\chi_V = \sum_{i=1}^{k} j_i \chi_{W_i}$

$\pi(G, V)$, $V = W_1^{j_1} \oplus W_2^{j_2} \oplus \cdots \oplus W_k^{j_k}$ $\begin{pmatrix} \boxed{W_1} & & \\ & \boxed{W_1} & \\ & & \ddots \\ & & & \boxed{W_k} \end{pmatrix}$

$$\langle \chi_W, \chi_V \rangle = \sum_{i=1}^{k} j_i \langle \chi_W, \chi_{W_i} \rangle = j(W).$$

$$\Big\| \quad \frac{1}{|G|} \sum_{g \in G} \chi_W(g) \cdot \overline{\chi_V(g)} = \frac{1}{|G|} |G| \cdot \dim W.$$

Claim: Write $V_{reg} = W_1^{j_1} \oplus \cdots \oplus W_k^{j_k}$, W_i distinct irreps.

Then $|G| = \sum_{i=1}^{k} j_i^2 = \sum_{i=1}^{k} (\dim W_i)^2$.

pf: $\frac{1}{|G|}|G|^2 = \langle \chi_V, \chi_V \rangle = \sum_{i=1}^{k} j_i \underbrace{\langle \chi_{W_i}, \chi_V \rangle}_{"j_i"}$.

Back to $V = L^2(G)$, $G = S_3$, $|G| = 6 = \sum j_i^2$.

$V = W_1 \oplus W_1^{\perp}$ $\quad$ $1^2 + \underline{1^2 + 2^2}$

$\chi_1 = \pi_1 : \begin{pmatrix} e \\ \{a\} \\ \{b\} \end{pmatrix} \mapsto \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$

$\mathbb{C} e_1$, $e_1 = (1,1,1,1,1,1)$, $\quad ab = ba^2 b$

$$a: \begin{pmatrix} e \\ a \\ a^2 \\ b \\ ab \\ a^2 b \end{pmatrix} \mapsto \begin{pmatrix} a \\ a^2 \\ e \\ a^2 b \\ b \\ ab \end{pmatrix} = \begin{pmatrix} 0 & 1 & 0 & & & \\ 0 & 0 & 1 & & & \\ 1 & 0 & 0 & & & \\ & & & 0 & 0 & 1 \\ & & & 1 & 0 & 0 \\ & & & 0 & 1 & 0 \end{pmatrix} \begin{pmatrix} e \\ a \\ a^2 \\ b \\ ab \\ a^2 b \end{pmatrix}$$

$$b: \begin{pmatrix} e \\ a \\ a^2 \\ b \\ ab \\ a^2 b \end{pmatrix} \mapsto \begin{pmatrix} b \\ ab \\ a^2 b \\ e \\ a \\ a^2 \end{pmatrix} = \begin{pmatrix} & & & 1 & & \\ & & & & 1 & \\ & & & & & 1 \\ 1 & & & & & \\ & 1 & & & & \\ & & 1 & & & \end{pmatrix} \begin{pmatrix} e \\ a \\ a^2 \\ b \\ ab \\ a^2 b \end{pmatrix}$$

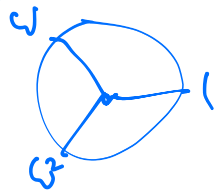
$$e_2 = (1,1,1,-1,-1,-1).$$

$$\pi_2(a)\, e_2 = e_2.$$

$$\pi_2(b)\, e_2 = -e_2.$$

$$\chi_2 \begin{pmatrix} e \\ \{a\} \\ \{b\} \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \\ -1 \end{pmatrix} \qquad W_2 = \mathbb{C} e_2.$$

$$\omega = -\frac{1}{2} + \frac{\sqrt{3}\, i}{2}, \qquad \omega^3 = 1$$



W_3 $\parallel$ $\langle e_3, e_4 \rangle$:

$$\begin{cases} e_3 = (1, \omega, \omega^2, 0, 0, 0) \\ e_4 = (0, 0, 0, 1, \omega, \omega^2) \end{cases}$$

$$\pi_3(a)\, e_3 = (\omega^2, 1, \omega, 0, 0, 0) = \omega^2 e_3.$$

$$\pi_3(b)\, e_3 = (0,0,0,1,\omega,\omega^2)$$

$$\pi_3(a).e_4 = (0,0,0,\omega,\omega^2,1) = \omega e_4.$$

$$\pi_3(b).e_4 = e_3.$$

$$\pi_3(a) \begin{bmatrix} e_3 \\ e_4 \end{bmatrix} = \begin{bmatrix} \omega^2 & 0 \\ 0 & \omega \end{bmatrix} \begin{bmatrix} e_3 \\ e_4 \end{bmatrix}$$

$$\pi_3(b) \begin{bmatrix} e_3 \\ e_4 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} e_3 \\ e_4 \end{bmatrix}$$

$$\chi_3 \begin{pmatrix} e \\ \{a\} \\ \{b\} \end{pmatrix} = \begin{pmatrix} 2 \\ -1 \\ 0 \end{pmatrix} \leftarrow$$

$$V_{reg} = W_1 \oplus W_2 \oplus W_3 \oplus \tilde{W}_3.$$

To find e_5, take

$$\begin{vmatrix} 1 & 1 & 1 \\ 1 & \omega & \omega^2 \\ i & j & k \end{vmatrix} = (\omega^2 - \omega,\ 1 - \omega^2,\ \omega - 1).$$

$$e_5 = (1, \omega^2, \omega, 0, 0, 0)$$

$$\langle e_3, e_5 \rangle = 1 \cdot 1 + \omega \cdot \omega + \omega^2 \cdot \omega^2 = 0.$$

σ_3: $e_\epsilon = (0, 0, 0, 1, \omega^2, \omega)$.

① Irred rep's.
② Decomp all $L^2(G)$.

One example of ② occurring in applications.

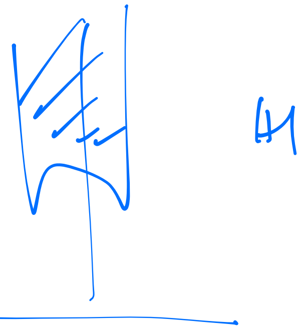
Let $\Gamma = SL_2(\mathbb{Z})$, $G = SL_2(\mathbb{R})$

$\mathcal{N}(T) := \#\left\{ \gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma : a^2 + b^2 + c^2 + d^2 < T^2 \right\}$

$ad - bc = 1$

$a^2+b^2+c^2+d^2 = \operatorname{tr}(\gamma \cdot \gamma^t) = \|\gamma\|^2$ "Frobenius norm".

$D = \Gamma \backslash G$



Let $f_T(g) := \mathbb{1}_{\|g\| < T} : G \to \mathbb{C}$.

Automorphize f_T to $F_T(g) := \sum_{\gamma \in \Gamma} f_T(\gamma g) : \Gamma \backslash G \to \mathbb{C}$

$F_T(\gamma g) = F_T(g) = F_T([g]) = F_T(\Gamma g)$.

$\boxed{F_T(e) = \mathcal{N}(T)}$.

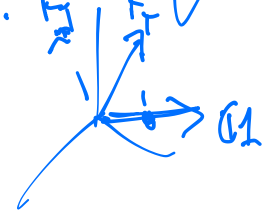
$F_T \in L^2(\Gamma \backslash G, \mu)$ — Haar meas on G.

Fact: $\operatorname{vol}(\Gamma \backslash G) < \infty$.

Want: decompose $\uparrow V$ into irreducibles

(or just some nice invariant subspaces),

& study projection of F_T to subspaces. $F_T^{\perp}$, F_T



$$V = \mathbb{C}\cdot 1 \oplus V^{\perp}.$$

$$F_T(g) = \underbrace{\langle F_T, 1\rangle}\cdot \underbrace{1(g)}_{1} + F_T^{\perp}(g)$$

$$= \frac{1}{\mathrm{vol}(\Gamma\backslash G)} \int_{\Gamma\backslash G} F_T(g)\cdot \bar{1}\cdot dg$$

Fix fund dom D, for $\Gamma\backslash G$

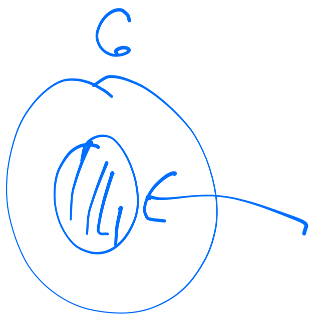
$$= \frac{1}{\mathrm{vol}(\Gamma\backslash G)} \int_D \sum_{\gamma\in\Gamma} f_T(\gamma g)\, dg$$

$$= \frac{1}{\mathrm{vol}} \sum_{\gamma\in\Gamma} \int_D f_T(\underbrace{\gamma g})\, dg$$

$h = \gamma g$

$$= \frac{1}{\mathrm{vol}} \sum_{\gamma\in\Gamma} \int_{\gamma D} f_T(h)\,dh = \frac{1}{\mathrm{vol}} \int_G f_T(h)\,dh$$

$$= \frac{\mathrm{vol}(B_T)}{\mathrm{vol}(\Gamma\backslash G)}$$



B_T : ball of radius T in G.

$$\sum_{\gamma\in\Gamma} f_T(\gamma) = F_T(e) = \frac{1}{\mathrm{vol}}\int_G f_T(h)\,dh + F_T^{\perp}(e)$$

analogue of Poisson summation

$$\sum_{n\in\mathbb{Z}} f(n) = \sum_{m\in\mathbb{Z}} \hat{f}(m) = \hat{f}(0) + \sum_{m\neq 0} \hat{f}(m)$$

$L^2(\mathbb{R}/\mathbb{Z})$.

Back to Dirichlet characters $\left(\subset L^2(\mathbb{Q}^\times\backslash\mathbb{A}^\times)\right)$.

$G = (\mathbb{Z}/N\mathbb{Z})^\times$, χ char : $G \to \mathbb{C}^\times$,

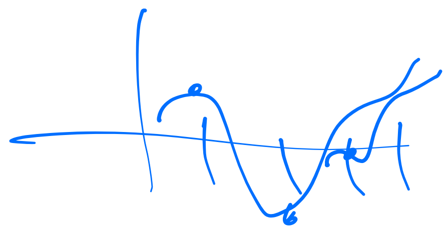
Lift χ from $(\mathbb{Z}/N\mathbb{Z})^\times$ to $\mathbb{Z}/N$.

Set $\chi(m) = 0$ if $(m, N) \neq 1$.

Lift χ to $\mathbb{Z}$. Want: "twisted Poisson summation formula":

$$\sum_{n \in \mathbb{Z}} f(n)\chi(n) = ?$$

To do: extend χ to $\mathbb{R}$.



On $\chi \in L^2(\mathbb{Z}/N\mathbb{Z}) = \bigoplus_{m \,(\mathrm{mod}\, N)} \mathbb{C}\, e_m$, $\quad e_m(a) = e^{2\pi i \frac{am}{N}}$.

$\quad\quad\quad\quad\; \ni f$

Decompose χ,

$$\boxed{\hat{\chi}(a) = \frac{1}{\sqrt{N}} \sum_{m \,(\mathrm{mod}\, N)} \chi(m)\, e^{2\pi i \frac{am}{N}}}$$

Exercise:

For any $f \in L^2(\mathbb{Z}/N\mathbb{Z})$, set $\hat{f}(a) = \frac{1}{\sqrt{N}} \sum_{m \,(\mathrm{mod}\, N)} f(m)\, e^{2\pi i \frac{am}{N}}$

then $f(m) = \frac{1}{\sqrt{N}} \sum_{a \,(\mathrm{mod}\, N)} \hat{f}(a)\, e^{-2\pi i \frac{am}{N}}$

$$\boxed{\chi(m) = \frac{1}{\sqrt{N}} \sum_{a \,(\mathrm{mod}\, N)} \hat{\chi}(a)\, e^{-2\pi i \frac{am}{N}}, \quad m \in \mathbb{R}.}$$