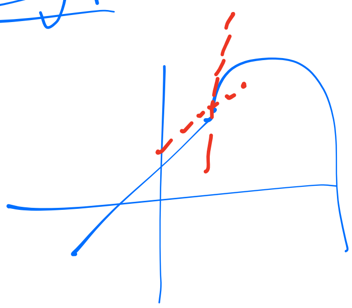
Back to Calc: f is **differentiable** at x if **derivative** $f'(x)$ exists. (at x).

Recall: $f'(x) = \lim\limits_{u \to x} \dfrac{f(u)-f(x)}{u-x} = \lim\limits_{h \to 0} \dfrac{f(x+h)-f(x)}{h}$.

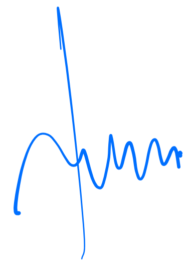
$\lim\limits_{u \to x}$ exists iff $\lim\limits_{u \to x^+}$ exists & $\lim\limits_{u \to x^-}$ exists & .

Eg: Two one-sided limits exist but differ.

"Corner"



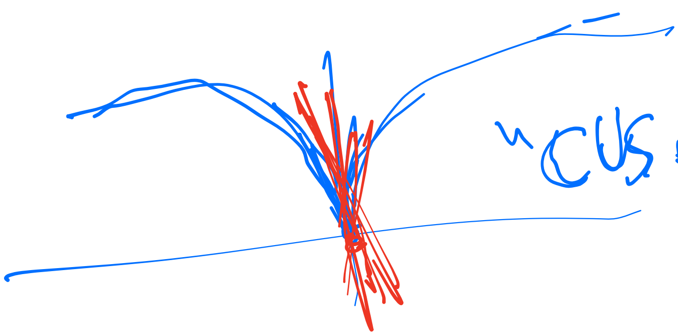
Eg: One sided limit could fail to exist



Eg: Both one-sided limits blow up in different direction

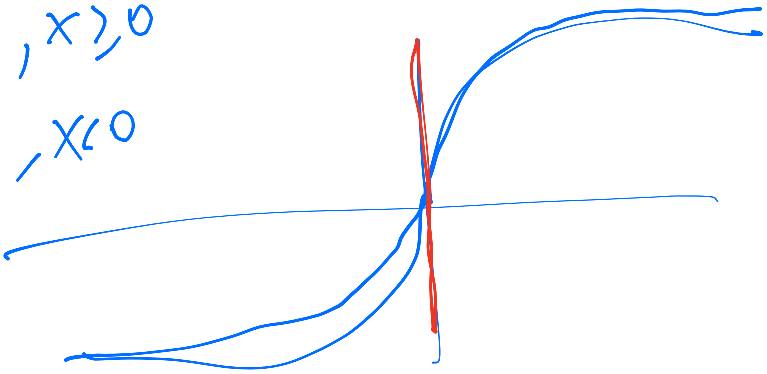
$y = \sqrt{x}$, $y' = \frac{1}{2} x^{-1/2} = \dfrac{1}{2\sqrt{x}} \to \infty$, as $x \to 0^+$

Eg: $y = \sqrt{|x|}$



"Cusp"

$$f(x) = \begin{cases} \sqrt{x}, & x \geq 0 \\ -\sqrt{-x}, & x < 0 \end{cases}$$



∞ is not a real number,

$\lim_{x \to 0} f'(x) = \infty$. This f is **not** differentiable at $x = 0$.

Let's compute derivative of exponentiation function: $f(x) = e^x$, $e = 2.71\ldots$

"Real" definition of $= \lim_{n \to \infty} \left(1 + \frac{1}{n}\right)^n$.

$$e^x = \frac{x^0}{0!} + \frac{x^1}{1!} + \frac{x^2}{2!} + \frac{x^3}{6} + \frac{x^4}{4!} + \frac{x^5}{5!} + \frac{x^6}{6!} + \ldots$$

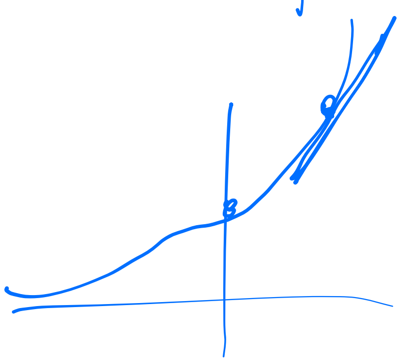
($1 = x^0$, $6 = 3!$)

$0! = 1$

$7! = 7 \cdot 6 \cdot 5 \cdot 4 \cdot 3 \cdot 2 \cdot 1$

$$\frac{d}{dx}(e^x) = 0 + 1 + \frac{1}{2}(2x) + \frac{1}{3!}(3x^2) + \frac{1}{4!}(4x^3) + \frac{1}{5!}(5x^4) + \ldots$$

$$= 1 + x + \frac{1}{2!}x^2 + \frac{1}{3!}x^3 + \frac{1}{4!}x^4 + \ldots = e^x$$



$y = e^x$

$$= \sum_{n=0}^{\infty} \frac{x^n}{n!}.$$

2^n: 1, 2, 4, 8, 16, 32, ---

1, 2, 4, 8, 16, ---

Trig Derivatives:

$$\frac{d}{dx}(\sin x) = \lim_{u\to x} \frac{\sin u - \sin x}{u - x} = \lim_{h\to 0} \frac{\sin(x+h) - \sin x}{h}.$$

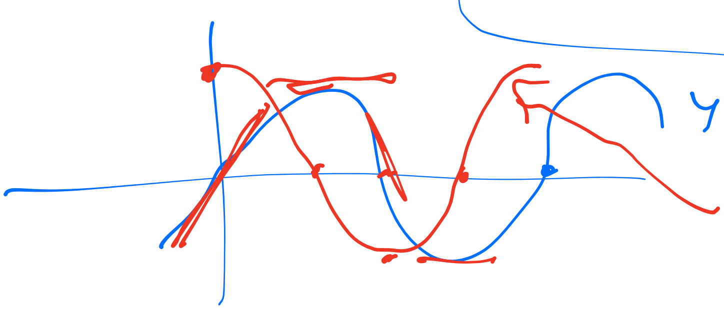
$$\sin(x+h) - \sin x = \sin x \cos h - \sin x + \cos x \sin h$$

$$\frac{\sin(x+h) - \sin x}{h} = \sin x \left(\frac{\cos h - 1}{h}\right) + \cos x \cdot \frac{\sin h}{h}$$

$$\lim_{h\to 0}: \quad (\sin x)' = \sin x \cdot 0 + \cos x \cdot 1$$

$$\frac{(\cos h - 1)}{h} \cdot \frac{(\cos h + 1)}{(\cos h + 1)} = \frac{-\sin h}{h} \cdot \frac{\sin h}{(\cos h + 1)} \to 0,$$

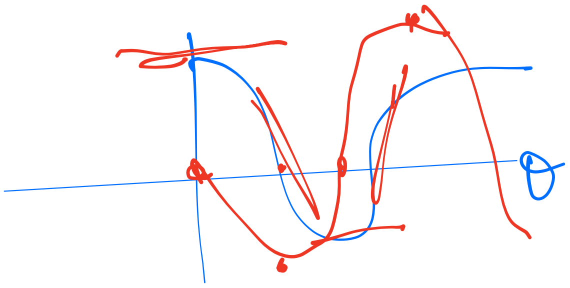
($\frac{\sin h}{h} \to 1$, $\sin h \to 0$, $\cos h + 1 \to 2$)



$y = \sin x$

$y' = \cos x.$

$$\frac{d}{d\theta} \cos\theta = -\sin\theta.$$



$y = \cos\theta$

$$\frac{d}{dz}\sin z = \cos z$$

$$\left(\frac{f}{g}\right)' = \frac{g\cdot f' - f\cdot g'}{g^2}$$

$$\frac{d}{dx}\sec t = ?$$

$$\frac{d}{dt}\tan t = \frac{d}{dt}\left(\frac{\sin t}{\cos t}\right) = \frac{\cos t\cdot\cos t - \sin t\cdot(-\sin t)}{\cos^2 t}$$

$$= \sec^2 t.$$

$$\left(\frac{1}{f}\right)' = \frac{-f'}{f^2}.$$

Quiz: $\frac{d}{dt}\left[\cot t\right] = \left[\frac{-\sec^2 t}{\tan^2 t}\right] = \frac{\sin t(-\sin t) - (\cos t\cdot\cos t)}{\sin^2 t}.$

$\cot t = \frac{\cos t}{\sin t}$

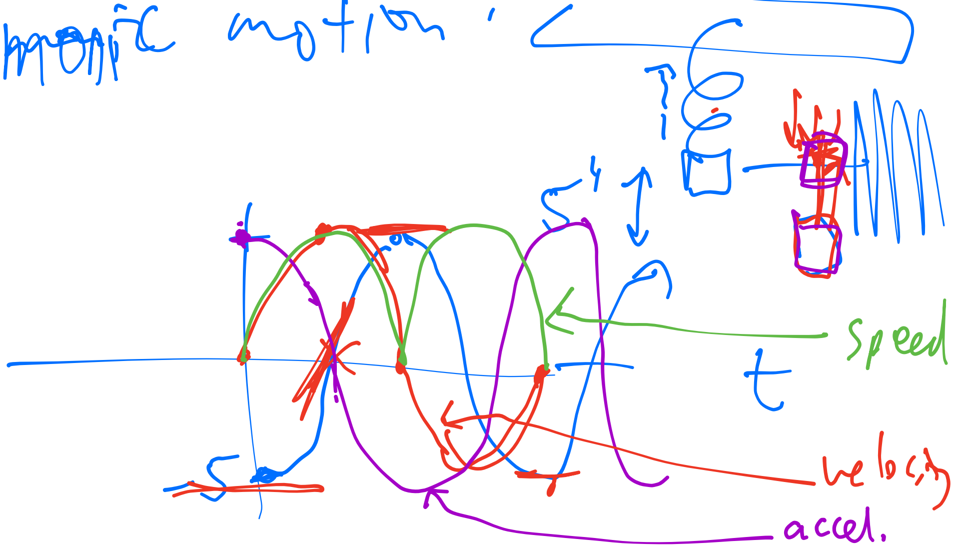
$-1 - \cot^2 t = \frac{-(\sin^2 t + \cos^2 t)}{\sin^2 t} = -\csc^2 t.$

$$\frac{d}{dx}e^{2x} = \frac{d}{dx}e^x\cdot e^x = e^x\cdot e^x + e^x\cdot e^x = 2\cdot e^{2x}.$$

Eg: "simple harmonic motion":

$s(t) = -5\cdot\cos t$

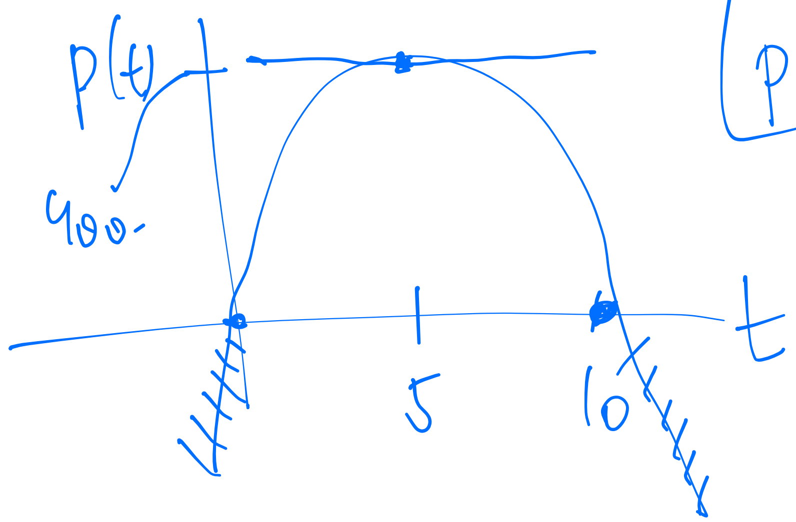
" position, time



velocity = $\frac{d}{dt}$ position, $v(t) = 5(\sin t)$, speed = |vel|

acceleration = $\frac{d}{dt}$ velocity $a(t) = 5\cos t$.

Ex: Throw a ball into air with initial velocity of 160 ft/sec. Gravity $-32\ ft/s^2$.



$p(t) = 160 \cdot t - \frac{32}{2} \cdot t^2$
$= 16t(10-t)$.

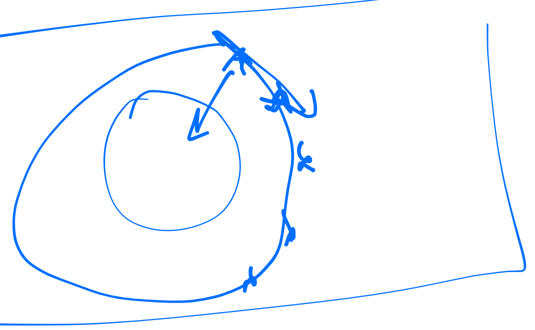
Q: How high will it go?

Q: Flight time?

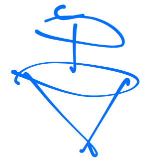
$v(t) = 160 - 16(2t) = 160 - 32t. = 0, \Rightarrow t = 5$

$p(5) = 16 \cdot 5(5) = 400$, $a(t) = -32$.

$\frac{d}{dt}$(acceleration) = jerk. jerk(t) = 0.



Eg! Factory makes spin tops



$C(x) =$ cost to make x spin tops.

$= x^3 - 6x^2 + 15x.$

$r(x) =$ revenue $=$ money from sell x tops.

$= x^3 - 3x^2 + 12x.$

Marginal cost at $x = 10$ tops

$$C'(x)\Big|_{x=10} = 3x^2 - 12x + 15\Big|_{x=10} = \$195.$$

marginal revenue $= r'(x)\Big|_{x=10} = 3x^2 - 6x + 12\Big|_{x=10} = \$252$

marginal net $= 252 - 195 = \$57.$

$$\frac{d}{dx} \sec x = \frac{d}{dx}\left(\frac{1}{\cos x}\right) = \frac{-(-\sin x)}{\cos^2 x} = \tan x \sec x.$$