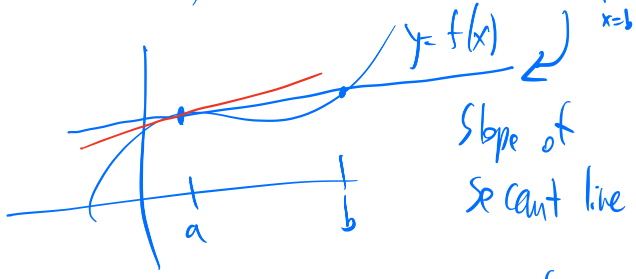
Recall: learned about rates of change, both on average: from $x=a$ to $x=b$



vs. instantaneous rate of change at $x=a$

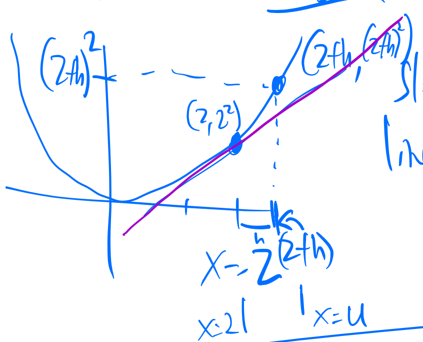
→ slope of tangent line $= \frac{0}{0}$.

$=$ lim of slopes of secant lines

Needed to master limits.

- continuity, continuous extensions
- Intermed. Val. Thm
- Squeeze Thm.
- asymptotes.

Back to tangent lines & their slopes. E.g.: $y = x^2 = f(x)$



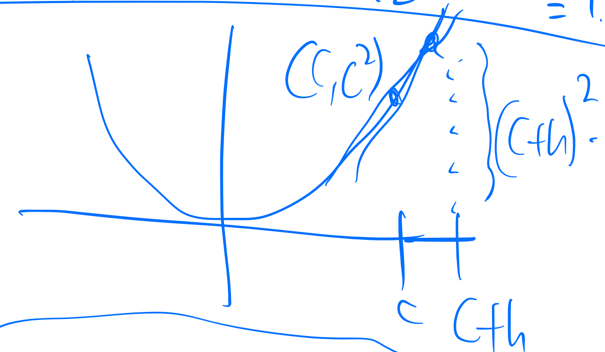
→ Slope $= \lim_{h\to 0} \frac{(2+h)^2 - 2^2}{(2+h)-2} = \lim_{h\to 0} \frac{f(2+h)-f(2)}{h}$

$= \lim_{h\to 0} \frac{\cancel{4} + 4\cancel{h} + h^{\cancel{2}} - \cancel{4}}{\cancel{h}\ 1} = \lim_{h\to 0} 4+h = 4.$

$= \lim_{u\to 2} \frac{f(u)-f(2)}{u-2} = \lim_{u\to 2} \frac{(u^2 - 2^2)}{\cancel{u-2}} = \lim_{u\to 2} u+2 = 4.$

$(u+2)(u-2)$

Instead of computing slope of tangent line at $x=2$, we can do same at $x=3$, or $x=c$.



Slope at $x=c$

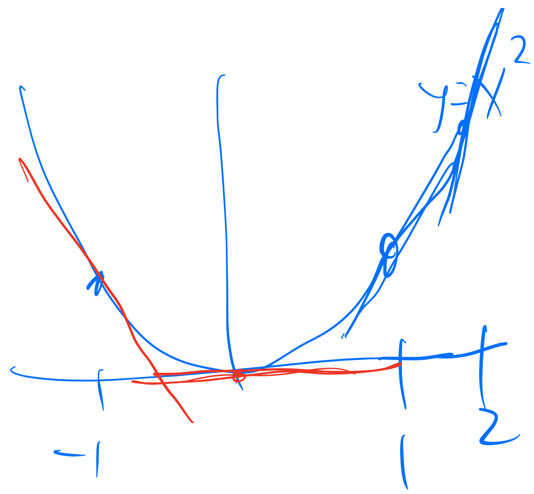
$= \lim_{h\to 0} \frac{(c+h)^2 - c^2}{c+h-c}$

$= \lim_{h\to 0} \frac{\cancel{c^2} + 2c\cancel{h} + h^{\cancel{2}} - \cancel{c^2}}{\cancel{h}\ 1} = \lim_{h\to 0} 2c+h$

$= 2c$

$$\lim_{u\to c} \frac{u^2-c^2}{u-c} = \lim_{u\to c} u+c = 2c$$

So: The slope of the tangent line to $y=x^2$ at $x=c$ is $2c$

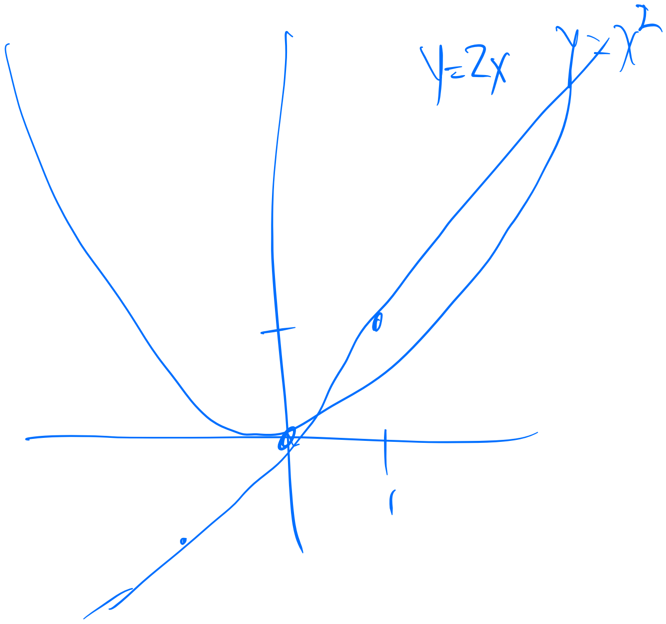


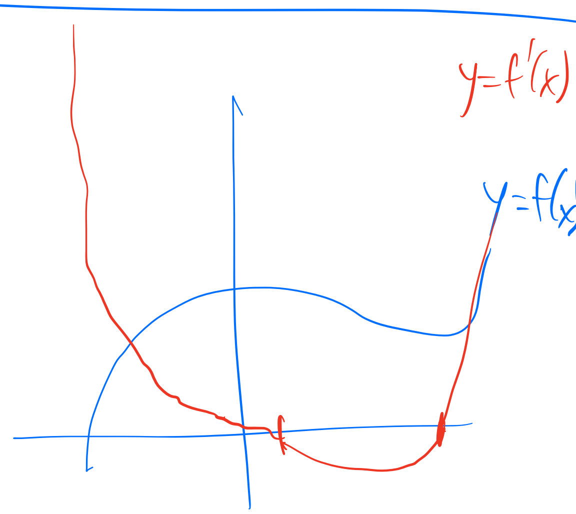
The function $g(x)=2x$ is the derivative of $f(x)=x^2$.

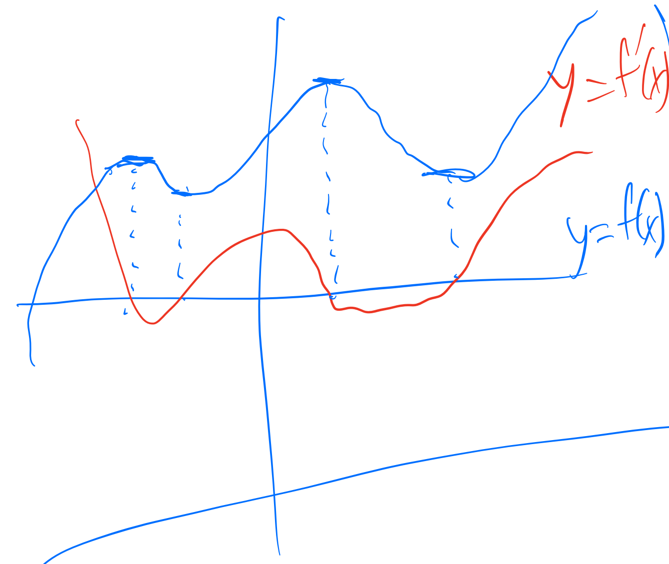
Notation: $f'(x)=2x$

"Derivative function":

$$\frac{d}{dx}(x^2) = \frac{d}{dx}(f(x)) = \frac{dy}{dx} = 2x$$

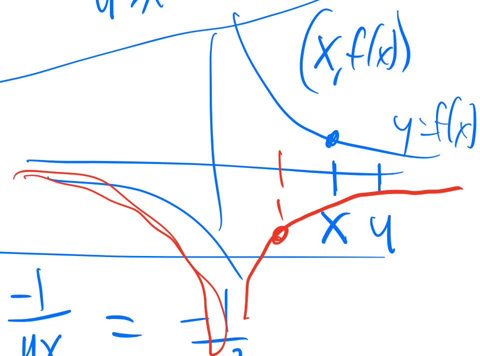






E.g.: $y = \frac{1}{x} = f(x)$.

$$\frac{df}{dx} = f'(x) = \lim_{u\to x} \frac{f(u)-f(x)}{u-x}$$



$$= \lim_{u\to x} \frac{\frac{1}{u} - \frac{1}{x}}{u-x} =$$

$$= \lim_{u\to x} \frac{\frac{x}{ux} - \frac{u}{ux}}{u-x} = \lim_{u\to x} \frac{\overset{-1}{\cancel{x-u}}}{ux} \cdot \frac{1}{\cancel{u-x}}$$

$$\lim_{u\to x} \frac{-1}{ux} = \frac{-1}{x^2}.$$

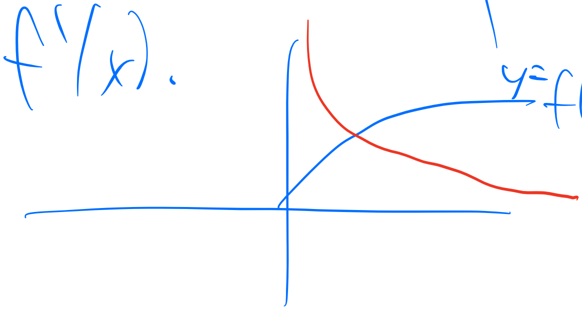
Alt: $\frac{\sqrt{u}-\sqrt{x}}{u-x} = \frac{\cancel{\sqrt{u}-\sqrt{x}}}{\cancel{(\sqrt{u}-\sqrt{x})}(\sqrt{u}+\sqrt{x})}$.

Quiz: Compute $f'(x)$ for $f(x)=\sqrt{x}$ & Draw $f'(x)$.

$$f'(x) = \lim_{u\to x} \frac{(\sqrt{u}-\sqrt{x})(\sqrt{u}+\sqrt{x})}{(u-x)(\sqrt{u}+\sqrt{x})}$$

$$= \lim_{u\to x} \frac{\overset{1}{\cancel{u-x}}}{\cancel{(u-x)}(\sqrt{u}+\sqrt{x})}$$

$$= \lim_{u\to x} \frac{1}{\sqrt{u}+\sqrt{x}} = \frac{1}{2\sqrt{x}}.$$



$$f(x) = \frac{x}{x-1}.$$

$$f'(x) = \lim_{h\to 0} \frac{f(x+h) - f(x)}{h}$$

$$= \lim_{h\to 0} \frac{\frac{x+h}{x+h-1} - \frac{x}{x-1}}{h}$$

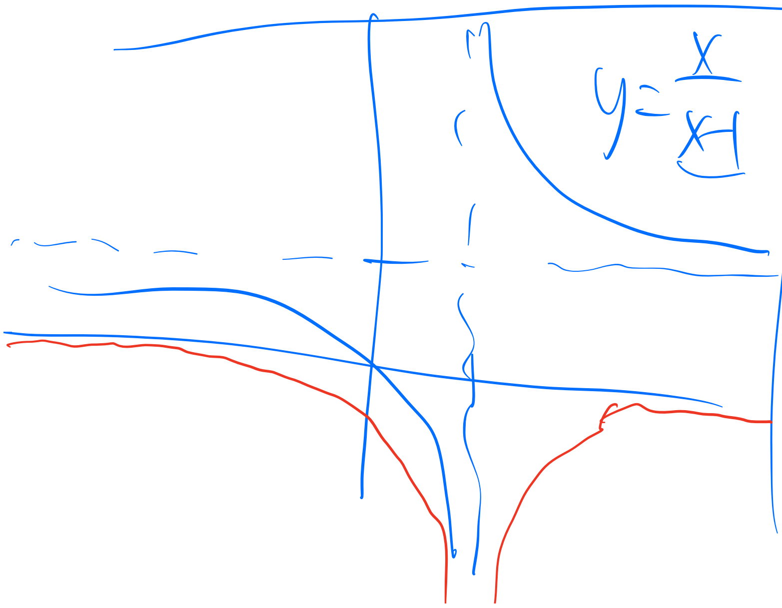
$$= \lim_{h\to 0} \frac{1}{h}\left[\frac{(x+h)(x-1)}{(x+h-1)(x-1)} - \frac{x(x+h-1)}{(x-1)(x+h-1)}\right]$$

$$= \lim_{h\to 0} \frac{1}{\cancel{h}}\left[\frac{\cancel{x^2} + \cancel{hx} - \cancel{x} - \overset{-1}{\cancel{h}} - (\cancel{x^2} + \cancel{hx} - \cancel{x})}{(x+h-1)(x-1)}\right]$$

$$= \lim_{h\to 0}\left[\frac{-1}{(x+h-1)(x-1)}\right]$$

$$= \frac{-1}{(x-1)^2}.$$

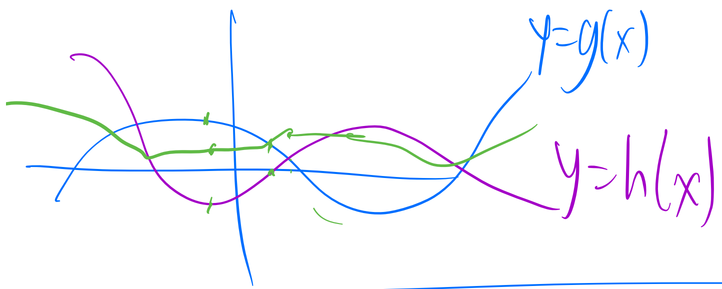
$$\cancel{\frac{(x-1)\cdot 1 - x}{(x-1)^2}}.$$



$f(x) = 3$, $f'(x) = 0$

$$f'(x) = \lim_{h\to 0} \frac{f(x+h) - f(x)}{h} = 0.$$

Derivative of a Constant $= 0$.



$f(x) = g(x) + h(x)$.

Can we learn anything about slope of f knowing slopes of g & h?

$$f'(x) = \lim_{u \to x} \frac{f(u) - f(x)}{u - x}$$

$$= \lim_{u \to x} \frac{g(u) + h(u) - [g(x) + h(x)]}{u - x}$$

$$= \lim_{u \to x} \left[\frac{g(u) - g(x)}{u - x}\right] + \left[\frac{h(u) - h(x)}{u - x}\right]$$

$$g'(x) + h'(x)$$

$$\frac{d}{dx}\left[x^2 + \frac{1}{x}\right]$$

$$= 2x + \frac{-1}{x^2}.$$