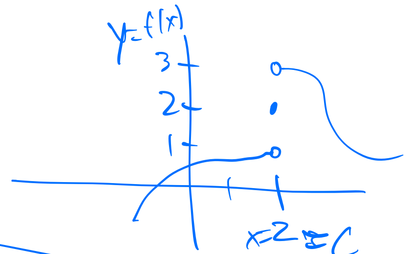
Recall: A limit does not care at all about the value of your function at the point.

E.g.:



$\lim_{x\to 2} f(x) = DNE$

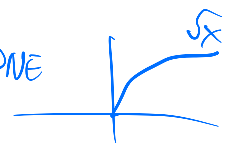
$\lim_{x\to 2^+} f(x) = 3$ $\qquad f(2)=2$

$\lim_{x\to 2^-} f(x) = 1$

For $\lim_{x\to c} f(x)$ to exist, need $\lim_{x\to c^+} f(x)$ to exist & $\lim_{x\to c^-} f(x)$ to exist & they agree.

Some Ways for $\lim_{x\to c^+} f(x)$ not exist:

(1) E.g.: $\lim_{x\to 0^-} \sqrt{x} = DNE$ (no values in domain).



(2) $\lim_{x\to 0^+} \frac{1}{x} = \infty$ (DNE).

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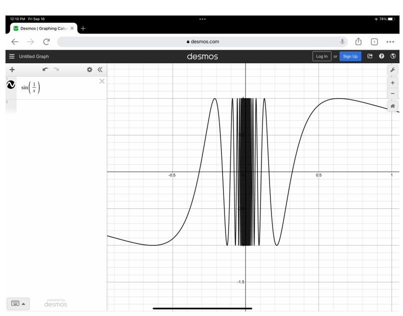
(3) Oscillatory behavior

E.g.: $y=\sin\left(\frac{1}{x}\right)$

Undefined at $x=0$. Near Zero:



$\lim_{x\to 0^+} \sin\left(\frac{1}{x}\right)$ DNE.

Aside: $\sin\left(\frac{1}{-x}\right) = -\sin\left(\frac{1}{x}\right)$,

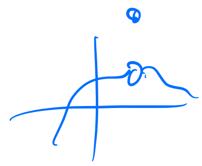
So $\sin\left(\frac{1}{x}\right)$ is odd.

Even this $\sin(\frac{1}{x})$, away from $x=0$, is continuous.

No breaks, draw without lifting a pen.

Def: f is continuous at $x=c$ if:

$\lim_{x\to c} f(x)$ exists & $f(c)$))



"Usual suspects", polynomials, trig sin cos, exp, logs all continuous in their domains.

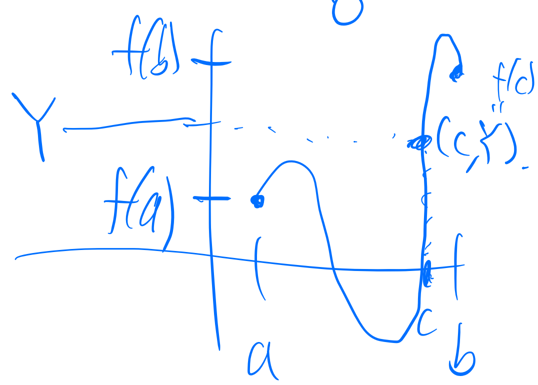
E.g.: $\lim_{x\to\frac{\pi}{2}} \cos\left(2x+\sin\left(\frac{3\pi}{2}+x\right)\right)$.

Because this is a composition of continuous functions, we don't need to know anything about the graph, we can plug in $x=\frac{\pi}{2}$

$= \cos\left(2\frac{\pi}{2}+\underbrace{\sin\left(\frac{3\pi}{2}+\frac{\pi}{2}\right)}_{0}\right) = -1.$

Intermediate Value Thm:

Say $f(x)$ is continuous on $[a,b]$. Then $\forall$ $Y\in[f(a),f(b)]$, $\exists$ $c\in[a,b]$ s.t. $f(c)=Y$.



$\in$ = "in"

$\forall$ = "for all"

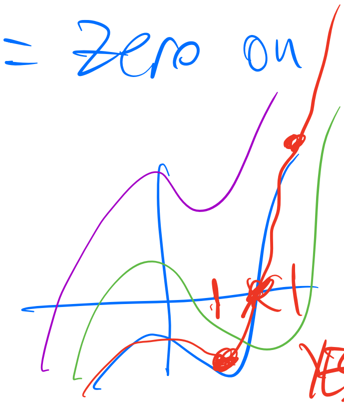
$\exists$ = "there exists"

s.t. = "such that"

E.g.: Does $f(x)=x^3-x-1$ have a root = zero on $[1,2]$.

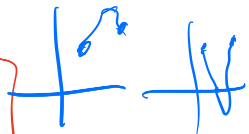
$f(1)=1^3-1-1=-1$

$f(2)=2^3-2-1=5$



YES BY IVT

If we had $f(1)=+$ & $f(2)=+$, IVT does not tell us anything about whether $f(x)=0$ between 1 & 2.



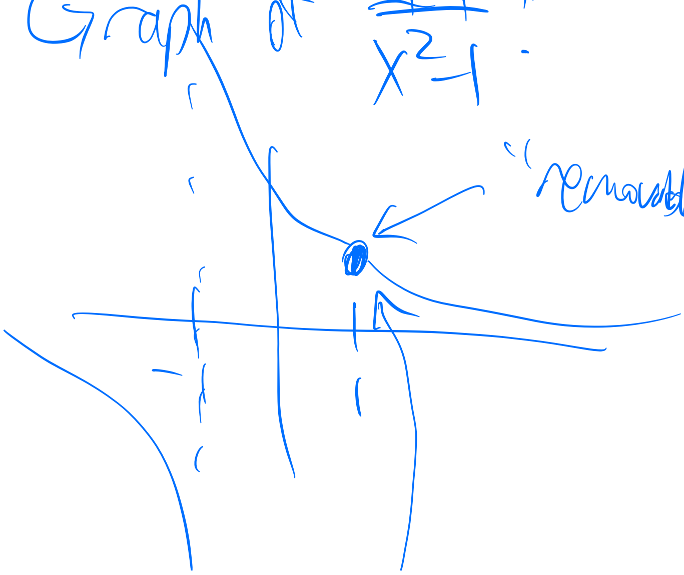
Remark: IVT can be used to prove that there is a zero but not to prove that there isn't a zero.

E.g.: $\lim_{x\to 1}\frac{x-1}{x^2-1}$

$= \frac{\cancel{x-1}}{\cancel{(x-1)}(x+1)} \to \frac{1}{x+1} = \frac{1}{2}$.

Cancel since $x \neq 1$.

Graph of $\frac{x-1}{x^2-1}$:

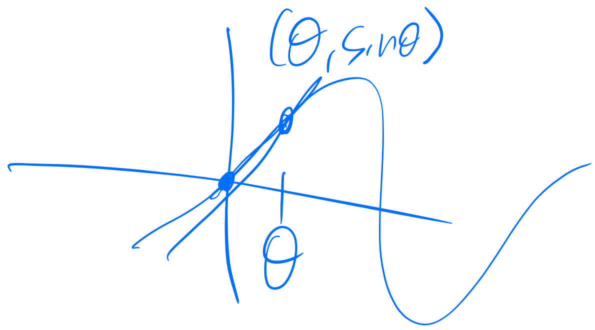


Def: if $\lim_{x\to c} f(x)$ exists but $f(c)$ ~~)~~ we can define "continuous extension"

E.g.: $g(x) = \begin{cases} \frac{x-1}{x^2-1}, & x \neq 1,\ x \neq -1 \\ \frac{1}{2}, & x = 1 \end{cases}$

$= \frac{1}{x+1}$

Last time: slope of sin near $x=0$.



Slope near 0 (or tangent line at $x=0$)

$$\lim_{\theta \to 0} \frac{\sin\theta}{\theta}$$

Saw:

$$\frac{1}{2}\cos\theta \cdot \sin\theta < \pi 1^2 \cdot \frac{\theta}{2\pi} < \frac{1}{2}\tan\theta$$

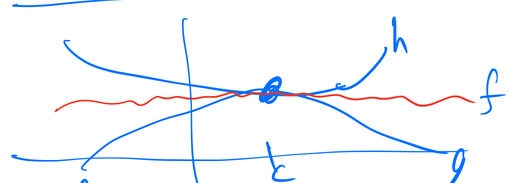
$$\sin\theta\cos\theta < \theta < \frac{\sin\theta}{\cos\theta}$$

$$\cos\theta < \frac{\theta}{\sin\theta} < \sec\theta.$$

$$\sec(\theta) > \frac{\sin\theta}{\theta} > \cos\theta$$

$\boxed{\text{By Sandwich Thm: } \lim_{\theta \to 0} \frac{\sin\theta}{\theta} = 1}$

Squeeze/Sandwich Thm:



If $g(x) \le f(x) \le h(x)$ near $x=c$ & $\lim_{x\to c} g(x) = \lim_{x \to c} h(x)$

$\Rightarrow \lim_{x \to c} f(x)$ exist &

So $\frac{\sin x}{x}$ has a continuous extension:

$$g(x) = \begin{cases} \frac{\sin x}{x}, & x \neq 0 \\ 1, & x = 0 \end{cases}$$