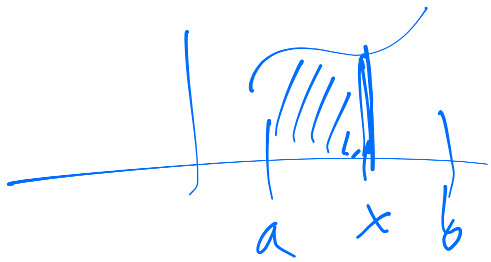
Final Exam: Fri 16th, 12-3 ARRIVE by 11:45

Last time: FTC (Fund. Thm of Calc)

Part 1: If f cont on a to b & $a \le x \le b$,

then $F(x) = \int_a^x f(t)\,dt$ is an antideriv of f,

$F'(x) = f(x)$



i.e. all f have antiderivs,

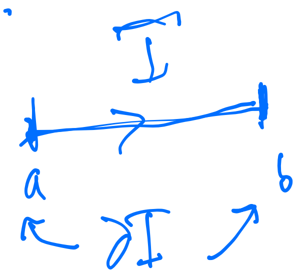
most F can't be expressed using poly/trig/exp/log.

Part 2: If we know an antideriv F, i.e. $F'(x) = f(x)$,

then $\int_a^b f(x)\,dx = F(b) - F(a)$.

Stokes
Green's

$$\int_I \partial F = \int_{\partial I} F$$



duality: boundary of funct & boundary of space ... —

Integration techniques: patterns for finding antiderivs

Eg: $\int_0^1 x\sqrt{x\sqrt[3]{x\sqrt[4]{x\sqrt[5]{x\sqrt[6]{\cdots}}}}}\,dx$

$e^x = 1+x+\frac{x^2}{2}+\frac{x^3}{3!}+\frac{x^4}{4!}+\cdots$

$\frac{d}{dx}e^x = 1+x+\frac{x^2}{2!}+\cdots = e^x$

$e = e^1 = 1+1+\frac{1}{2}+\frac{1}{3!}+\frac{1}{4!}+\frac{1}{5!}+\cdots$

$= \int_0^1 x^1 \cdot x^{1/2} \cdot x^{\frac{1}{2\cdot 3}} \cdot x^{\frac{1}{2\cdot 3\cdot 4}} \cdot x^{\frac{1}{2\cdot 3\cdot 4\cdot 5}} \cdots \, dx$

$= \underbrace{\int_0^1 x^{e-1}\,dx}_{\text{definite integral = area}} = \frac{x^e}{e}\Big|_{x=0}^{1} = \frac{1}{e} - 0 = \boxed{\frac{1}{e}}.$

Power Rule: $\frac{d}{dx}x^\alpha = \alpha x^{\alpha-1} \to \int x^\alpha dx = \frac{x^{\alpha+1}}{\alpha+1} + C.$

Every deriv rule $\to$ integral "rule" = short cut.

Exp Rule: $\frac{d}{dx}a^x = a^x \cdot \ln a, \quad \int a^x dx = \frac{a^x}{\ln a} + C.$

indef integral = antideriv

Chain Rule: $\frac{d}{dx}\left[F(g(x))\right] = F'(g(x)) \cdot g'(x). \to$ "u-substitution"

Eg: $\int \underbrace{(\underbrace{x^3+x}_{u})^{21} \underbrace{(3x^2+1)\,dx}_{du}}$ Notice if $u = x^3+x$

$du = 3x^2dx + 1\cdot dx$

$= (3x^2+1)\,dx$

$= \int u^{21}\,du = \frac{u^{22}}{22} + C, = \boxed{\frac{(x^3+x)^{22}}{22} + C}$

Eg: $\int \sqrt{2x+1}\,dx$

$u = 2x+1$

$du = 2\,dx$

$\frac{1}{2}du = dx$

$= \frac{1}{2}\int u^{1/2}\,du = \frac{1}{2}\cdot\frac{2}{3}u^{3/2} + C$

$= \frac{1}{3}(2x+1)^{3/2} + C.$

Always check by differentiating!!!

$\frac{d}{dx}(\;) = \frac{1}{\cancel{3}}\cdot\frac{\cancel{3}}{\cancel{2}}(2x+1)^{1/2}(\cancel{2})$ ✓

Eg: $\int \sec^2(5x+1)\,dx$

$u = 5x+1$

$du = 5\,dx$

$\frac{1}{5}du = dx$

$= \frac{1}{5}\int \sec^2 u\,du$

$= \frac{1}{5}\tan u + C = \frac{1}{5}\tan(5x+1) + C.$

Quiz: $\int e^{x^3} x^2\,dx$

$u = x^3$

$\to du = 3x^2\,dx$

$\frac{1}{3}du = x^2\,dx$

$= \frac{1}{3}\int e^u\,du = \frac{1}{3}e^{x^3} + C.$

For u-sub to work, need to convert all x & dx's into u & du's.

Eg: $\int \tan x \, dx = \int \frac{\sin x \, dx}{\cos x}$

$u = \cos x$

$du = -\sin x \, dx$

$-du = \sin x \, dx$

$= -\int \frac{du}{u} = -\int \frac{1}{u} \cdot du = -\ln|u| + C$

$= \underbrace{-\ln|\cos x| + C}_{} = \ln|\cos x|^{-1} + C = \ln|\sec x| + C.$

$\frac{d}{dx}\left[\ln|\sec x|\right] = \frac{1}{\cancel{\sec x}} \cdot \cancel{\sec x} \tan x = \tan x$

Eg: $\int \frac{2z}{\sqrt[3]{z^2+1}} \, dz$

$u = z^2+1$

$du = 2z \, dz$

$= \int u^{-1/3} \, du = \frac{3}{2} \cdot u^{2/3} + C = \boxed{\frac{3}{2}(z^2+1)^{2/3} + C.}$

OR: $u = \sqrt[3]{z^2+1}$, $du = \frac{1}{3}\underbrace{(z^2+1)^{-2/3}}_{u^{-2}} \cdot 2z \, dz.$

$= (z^2+1)^{1/3}$

$3u^2 \, du = 2z \, dz$

$= \int \frac{3u^{\cancel{2}} \, du}{\cancel{u}} = 3\frac{u^2}{2} + C = \boxed{\frac{3}{2}(z^2+1)^{2/3} + C}$

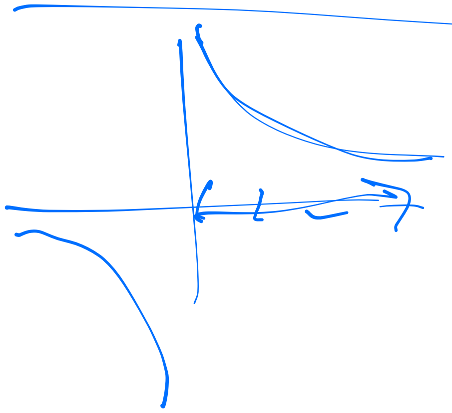
Eg: $\int \sec x \, dx = \int \sec x \cdot \frac{(\sec x + \tan x)}{\sec x + \tan x} \, dx$

$$= \int \frac{(\sec^2 x + \sec x \tan x)}{\sec x + \tan x} \, dx$$

$u = \sec x + \tan x$

$du = (\sec x \tan x + \sec^2 x) \, dx$

$$= \int \frac{1}{u} \, du = \ln|u| + C = \ln|\sec x + \tan x| + C.$$



$\int 1/x \, dx = \ln x$, on $x > 0$

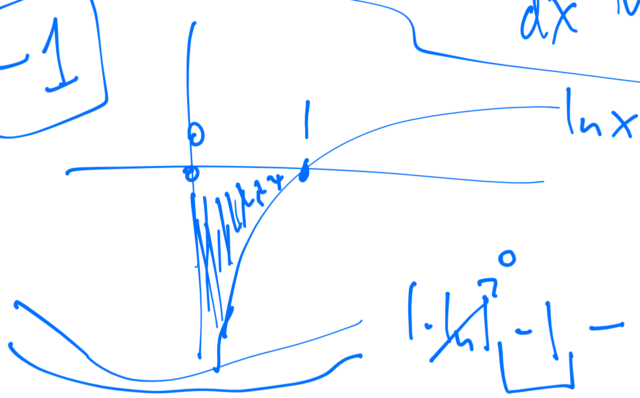
$-1/x$ on $x < 0$

$\boxed{\ln|x|}$ $= \ln(-x)$

$\frac{d}{dx} \ln(-x) = \frac{1}{-x}(-1) = \frac{1}{x}.$

Aside:

$\int_0^1 \ln x \, dx = \boxed{-1}$



$= x \ln x - x \Big|_0^1$ $\quad 1 \cdot \cancel{\ln 1}^{\,0} - 1 - (\cancel{\lim_{x \to 0} x \ln x}^{\,0} - 0) = -1$

$$\frac{d}{dx}(x \ln x - x) = x \cdot \frac{1}{x} + 1 \cdot \ln x - 1 = \ln x$$

$$\lim_{x \to 0} \frac{\ln x}{1/x} \stackrel{(L'H)}{=} \lim_{x \to 0} \frac{\frac{1}{x}}{-1/x^2} = \lim_{x \to 0} -\frac{1}{x} \cdot \frac{x^2}{1} = -0 = 0.$$

Eg: $\int \sin^2 x \, dx$

$\cos 2x = \cos^2 x - \sin^2 x$

$= 1 - 2\sin^2 x$

$\sin^2 x = \dfrac{1-\cos 2x}{2}$

$= \int \left(\frac{1}{2} - \frac{\cos 2x}{2}\right) dx$

$= \frac{x}{2} - \frac{1}{2} \frac{\sin(2x)}{2} + C.$

Check: $\frac{d}{dx}\left[\frac{x}{2} - \frac{1}{4}\sin(2x)\right] = \frac{1}{2} - \frac{1}{4}\cos(2x)\cdot 2$

Definite integrals with u-subs:

Eg: $\int_{-1}^{1} 3x^2 \sqrt{x^3+1} \, dx$,

let: $u = x^3+1$

$du = 3x^2 dx$

side calc of antideriv

$\boxed{\int 3x^2 \sqrt{x^3+1}\, dx = \int u^{1/2} du = \frac{2}{3} u^{3/2} + C = \frac{2}{3}(x^3+1)^{3/2} + C}$

$\frac{2}{3}(x^3+1)^{3/2} \Big|_{-1}^{1} = \frac{2}{3}\cdot 2^{3/2} - \left[\frac{2}{3}\cdot 0\right] = \boxed{\frac{2}{3}\cdot 2^{3/2}}$

OR

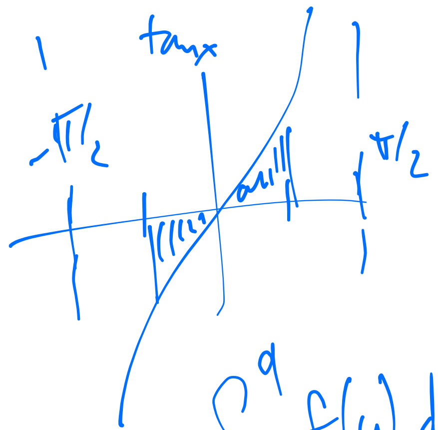
$\int_{\cancel{-1}\, 0}^{\cancel{1}\, 2} u^{1/2} du = \frac{2}{3} u^{3/2} \Big|_0^2 = \boxed{\frac{2}{3}\cdot 2^{3/2} - 0}$

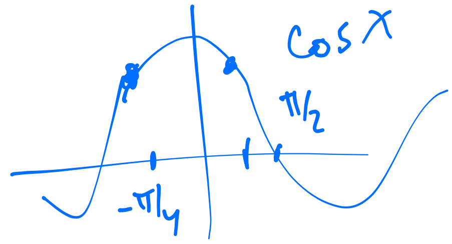
When $x=-1$, $u=0$

When $x=1$, $u=2$

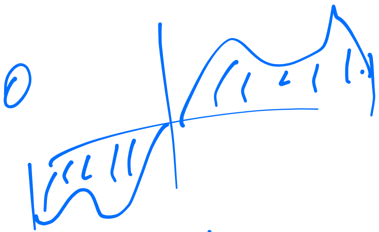
Eg: $\displaystyle\int_{-\pi/4}^{\pi/4} \tan x \, dx = \int_{-\pi/4}^{\pi/4} \frac{\sin x}{\cos x} \, dx$, $\quad u = \cos x$, $\; du = -\sin x \, dx$

$$= -\int_{\sqrt{2}/2}^{\sqrt{2}/2} \frac{du}{u} = \boxed{0}$$



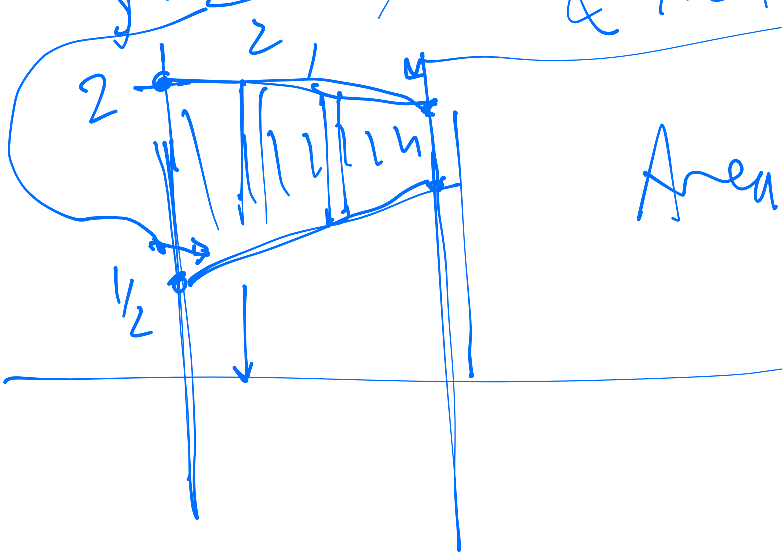


Thm: If f is odd, $\displaystyle\int_{-a}^{a} f(u)\,du = 0$

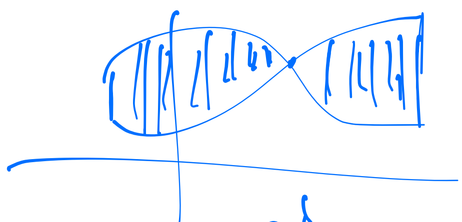


If f is even, $\displaystyle\int_{-a}^{a} f(u)\,du = 2\int_{0}^{a} f(\theta)\,d\theta$

Eg: Find area bdd by: $y = 2e^{-x} + x$, $y = \frac{e^x}{2}$ $\quad x = 0$ & $x = 1$

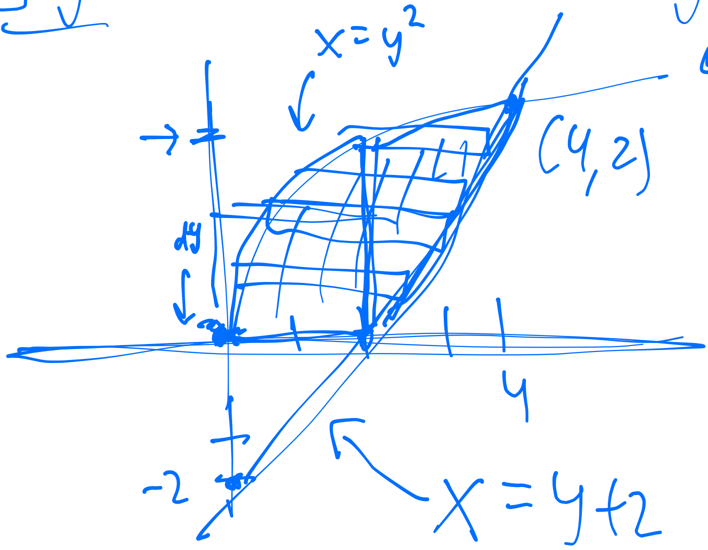


$$\text{Area} = \int_0^1 \left(2e^{-x} + x - \frac{e^x}{2}\right) dx.$$



Area between f & g from a to b: $\displaystyle\int_a^b |f(x) - g(x)| \, dx$

Eg: Find Area between $y=\sqrt{x}$, $y=0$, & $y=x-2$



Two ideas:

Idea 1: $\int_0^2 (\sqrt{x}-0)\,dx + \int_2^4 \left(\sqrt{x}-(x-2)\right)dx$

Idea 2: $\int_0^2 (y+2-y^2)\,dy$

$$= \frac{y^2}{2}+2y-\frac{y^3}{3}\Big|_0^2 = \frac{4}{2}+2\cdot2-\frac{8}{3}-0 = \boxed{\frac{10}{3}}$$

$$= \frac{2}{3}x^{3/2}\Big|_0^2 + \left(\frac{2}{3}x^{3/2}-\frac{x^2}{2}+2x\right)\Big|_2^4$$

$$= \cancel{\frac{2}{3}2^{3/2}} + \left[\frac{2}{3}\underbrace{4^{3/2}}_{8} - \underbrace{\frac{4^2}{2}}_{\cancel{-8}} + \cancel{+8}\,\underbrace{2\cdot4}_{} - \underbrace{\left(\cancel{\frac{2}{3}2^{3/2}} - \frac{2^2}{2} + 2\cdot2\right)}_{-2}\right]$$

$$\boxed{= \frac{10}{3}}$$