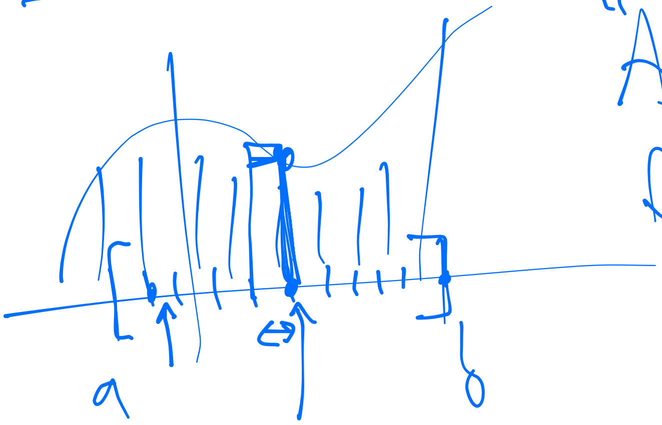
Last time: f cont. function on $[a,b]$,

"Area" under curve $y=f(x)$ from a to b:



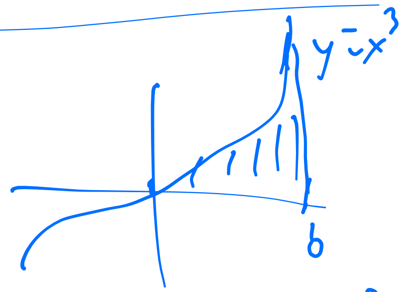
$$\int_a^b f(x)\,dx = \lim_{N\to\infty} \underbrace{\sum_{k=1}^{N} \left(\frac{b-a}{N}\right) f\left(a+k\frac{(b-a)}{N}\right)}_{\text{Riemann Sum.}}$$

right endpt $\to a + k\cdot\frac{(b-a)}{N}$

left endpt $\to k-1$

midpt: $a + (k-1)\frac{(b-a)}{N} + \frac{1}{2}\frac{(b-a)}{N} = a + \left(k-\frac{1}{2}\right)\frac{(b-a)}{N}$

Recall: $\int_0^b x^3\,dx = \boxed{\frac{b^4}{4}}$

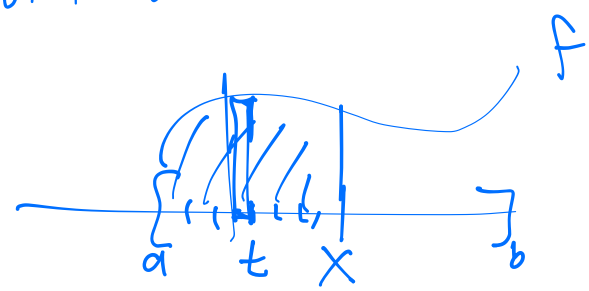


$$\sum_{k=1}^{N} k = \frac{N(N+1)}{2}, \quad \sum_{k=1}^{N} k^2 = \frac{N(N+1)(2N+1)}{6}, \quad \sum_{k=1}^{N} k^3 = \left[\frac{N(N+1)}{2}\right]^2.$$

Fundamental Thm of Calculus:

Idea: Consider not total area from a to b

but a running area:



Given f cont on $[a,b]$ & $a \le x \le b$. Let

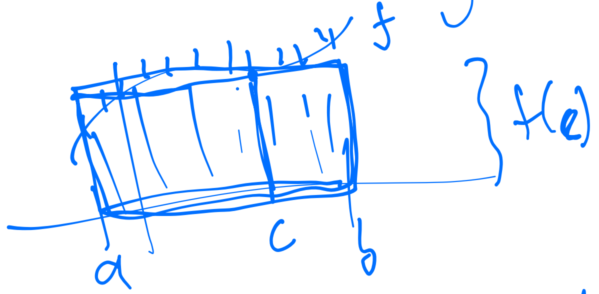
$F(x) := \int_a^x f(t)\,dt$ Idea: Compute

$\boxed{\frac{d}{dx} F(x)}$

Mean of $1, 2, 7, 12, -3, 14$,

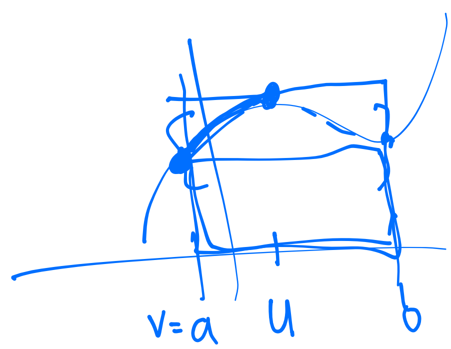
Add them up, divide by how many.

Mean of f on $[a,b]$ $:= \frac{1}{b-a}\int_a^b f(x)\,dx$



MVT for Integrals: If f cont on $[a,b]$, then $\exists\, c \in (a,b)$ s.t. $\frac{1}{b-a}\int_a^b f(x)\,dx = f(c)$.

$\iff \int_a^b f(x)\,dx = f(c)\cdot(b-a)$



Pf: f cont on $[a,b]$ then

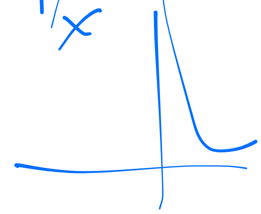
Extreme Val Thm: $\exists\, u$ s.t. $f(u) \ge f(x)\ \forall x \in [a,b]$

& $\exists\, v$ s.t. $f(v) \le f(x)\ \forall x \in [a,b]$

$\boxed{y=1}$

$$f(v)(b-a) \le \int_a^b f(x)\,dx \le f(u)\cdot(b-a)$$

On $[u,v]$ (or $[v,u]$), $(0,1)$ $1/x$

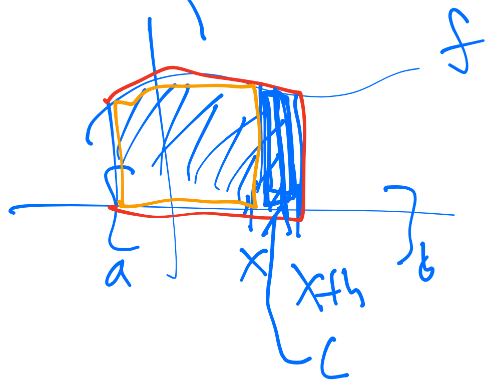


$$f(v) \le \underbrace{\frac{1}{b-a}\cdot\int_a^b f(x)\,dx} \le f(u)$$

By $\underline{\text{IVT}}$ ($\underline{\text{Intermediate Val Thm}}$) $\exists\, c$ s.t.

$f(c) =$ Avg val.

Back to Fund Thm: $F(x) = \int_a^x f(t)\,dt$



Compute $\frac{d}{dx}F(x)$,

$$= \lim_{h\to 0} \frac{F(x+h)-F(x)}{h}$$

$$= \lim_{h\to 0} \frac{1}{h}\left[\int_a^{x+h} f(t)\,dt - \int_a^x f(t)\,dt\right]$$

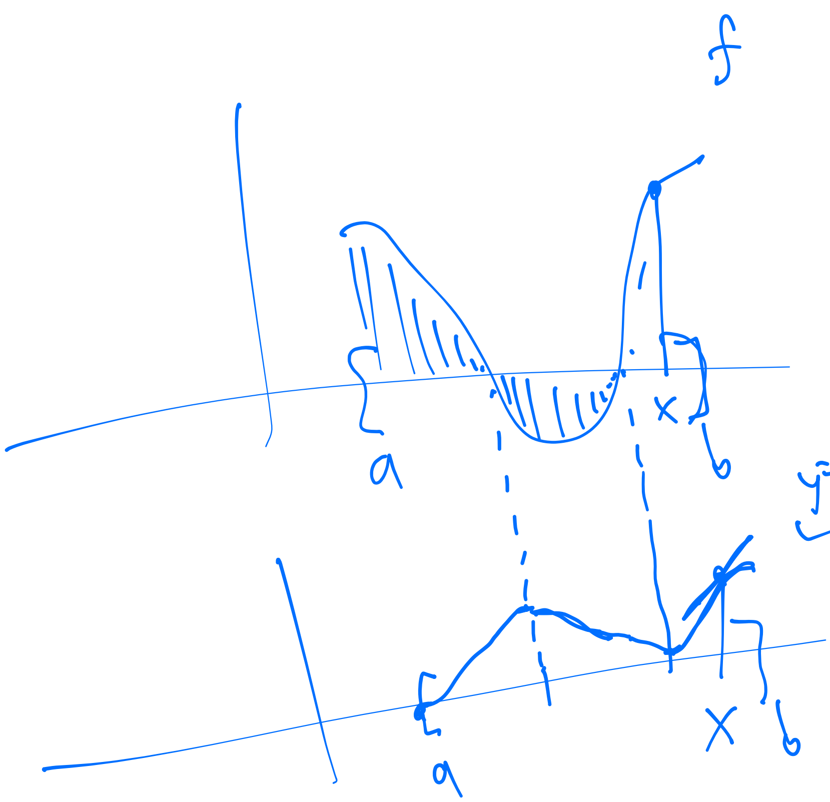
$$= \lim_{h\to 0} \underbrace{\frac{1}{h}\left[\int_x^{x+h} f(t)\,dt\right]}_{\text{Avg val of } f \text{ on } [x,x+h]} = \lim_{h\to 0} f(c_h) = f(x)$$

By MVT for integrals, $\exists\ c_h \in [x, x+h]$ s.t.

Since $x \le c_h \le x+h$, $\lim_{h\to 0} c_h = x$ (Squeeze thm).

__So:__ $\dfrac{d}{dx}\displaystyle\int_a^x f(t)\,dt = f(x)$

__Another viewpoint:__



$\underline{y = F(x)} = \displaystyle\int_a^x f(t)\,dt$

Again: FTC (Part I): $\frac{d}{dx}\int_a^x f(t)\,dt = f(x)$

FTC (Part II): If F is an antideriv of f, that is, $F'(x) = f(x)$, then (evaluation thm):

$$\int_a^b f(x)\,dx = F(b) - F(a).$$

$\boxed{G(a) = 0.}$

Pf: Look at $G(x) = \int_a^x f(t)\,dt$.

$$\frac{d}{dx}G(x) \overset{\text{FTC I}}{=} f(x) = \frac{d}{dx}F(x).$$ Both F & G are antiderivs of f. $\Rightarrow$ $F(x) = G(x) + C$.

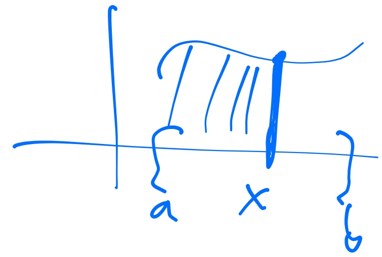
There exists some C const. s.t.

$$= G(b) - G(a)$$

$$= F(b) - F(a) = G(b) + \cancel{C} - (G(a) + \cancel{C})$$

Full Fund Thm of Calculus:

① $\frac{d}{dx}\int_a^x f(t)\,dt = f(x)$

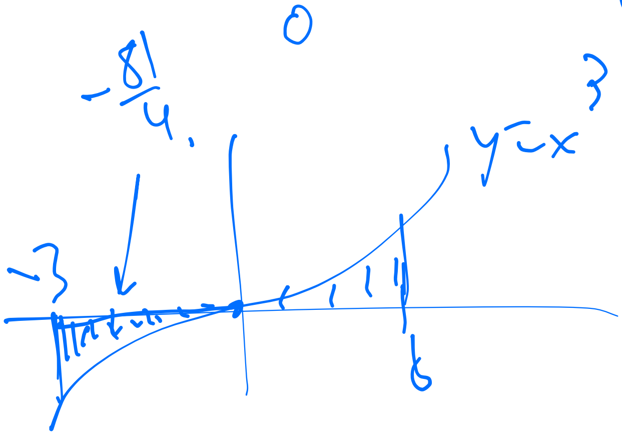


② If $F' = f$, then

$$\int_a^b f(t)\,dt = F(b) - F(a) = F\Big|_a^b$$

Ex: $f(x) = x^3$, $F(x) = \frac{x^4}{4}$.

$$\int_0^b x^3\,dx = \frac{x^4}{4}\Big|_0^b = \frac{b^4}{4} - \cancel{\frac{0^4}{4}}.$$



If $b<0$ ~~(0,-3)~~

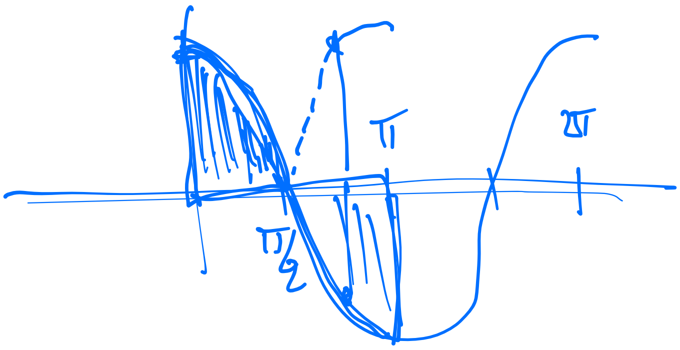
$$\int_0^{-3} x^3\,dx = \frac{81}{4}.$$

$$= -\int_{-3}^{0} x^3\,dx$$

$$\int_b^a f(t)\,dt = -\int_a^b f(t)\,dt.$$

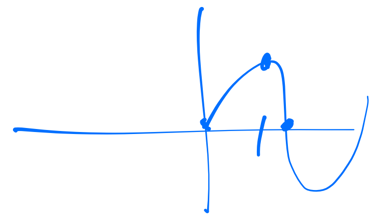
$$\int_0^{\pi} \cos x\,dx = \sin x\Big|_0^{\pi} = \sin\pi - \sin 0 = 0.$$

Q: What is the area between $y = \cos x$ & $y = 0$ (x-axis) from 0 to π.



$$\int_0^{\pi} |\cos x|\,dx = \int_0^{\pi/2} \cos x\,dx + \int_{\pi/2}^{\pi} (-\cos x)\,dx$$

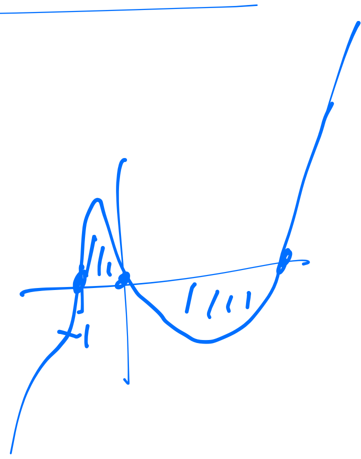
$$= \sin x\Big|_0^{\pi/2} + (-\sin x)\Big|_{\pi/2}^{\pi}$$



$$= \underbrace{\sin \tfrac{\pi}{2} - \sin 0} + \left[-\sin \pi - \left(-\sin \tfrac{\pi}{2}\right)\right]$$

$$= 1 - 0 + [0 + 1] = 2.$$

Ex: Area between x-axis $(y=0)$ and $y = x^3 - x^2 - 2x$ on $-1 \le x \le 2$.



$$\int_{-1}^{2} |x^3 - x^2 - 2x|\,dx$$

$$x^3 - x^2 - 2x$$
$$= x(x^2 - x - 2)$$
$$= x(x-2)(x+1)$$

$$= \int_{-1}^{0} (x^3 - x^2 - 2x)\,dx + \int_0^2 (-x^3 + x^2 + 2x)\,dx$$

$$= \left(\frac{x^4}{4} - \frac{x^3}{3} - x^2 + \cancel{x}\right)\Bigg|_{-1}^{0} + \left(\underbrace{-\frac{x^4}{4} + \frac{x^3}{3} + x^2}\right)\Bigg|_{0}^{2}$$

$$= 0 - \left(\frac{(-1)^4}{4} - \frac{(-1)^3}{3} - (-1)^2\right) + \left(-\frac{2^4}{4} + \frac{2^3}{3} + 2^2 - 0\right).$$

Ex: $\frac{d}{dx} \underbrace{\int_1^{x^3} \underbrace{e^t \log t \cos t}\, dt} = ?$.

let $F(x) = \int_1^x e^t \log t \cos t \, dt$

$F'(x) = e^x \log x \cos x$.

$\frac{d}{dx}\left(F(x^3)\right) = F'(x^3) \cdot 3x^2$,

$$\boxed{= \left(e^{x^3} \cdot \log x^3 \cdot \cos(x^3)\right) \cdot 3x^2}$$