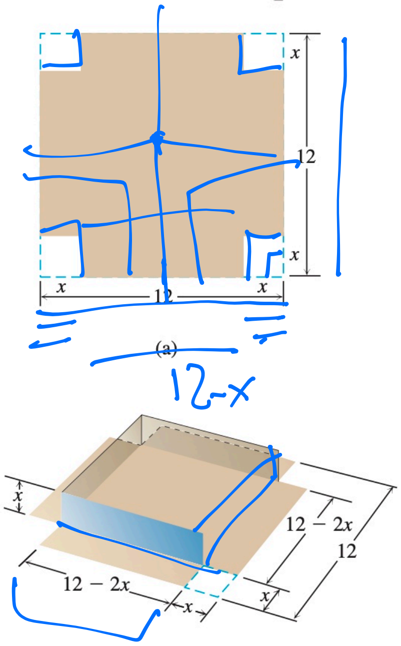
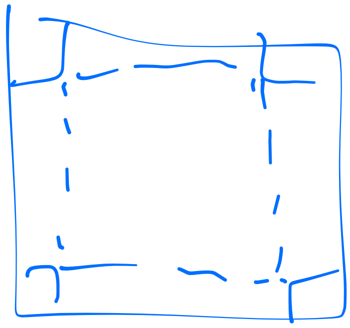
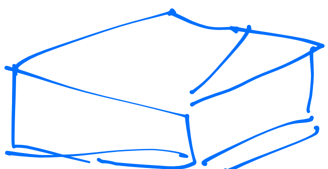
EXAMPLE 1 An open-top box is to be made by cutting small congruent squares from the corners of a 12-in.-by-12-in. sheet of tin and bending up the sides. How large should the squares cut from the corners be to make the box hold as much as possible?



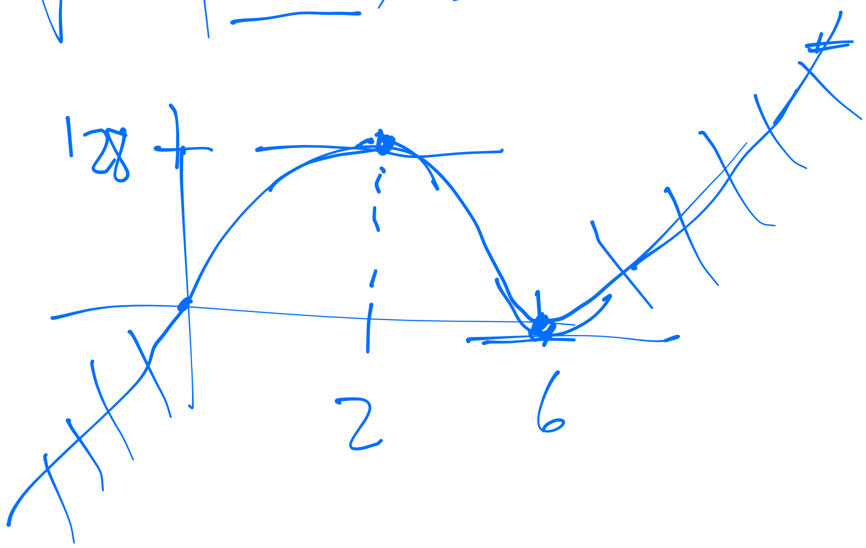


maximize volume
with constraint
12"×12" tin.



8 8

$$V = (12-2x)^2 \cdot x = 144x - 48x^2 + 4x^3$$



$$V' = 144 - 96x + 12x^2 = 12(x^2 - 8x + 12)$$

$$= 12(x-6)(x-2)$$

$x = 2''$, $V_{max} = 128 \text{ in}^3$

EXAMPLE 2 You have been asked to design a one-liter can shaped like a right circular cylinder (Figure 4.38). What dimensions will use the least material?

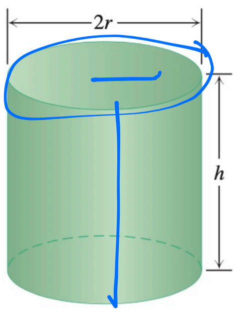
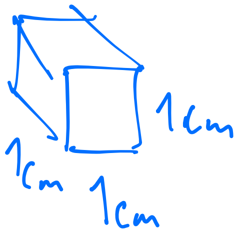


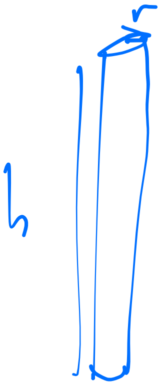
FIGURE 4.38 This one-liter can uses the least material when $h = 2r$ (Example 2).

Minimize material

Constant: $V = 1\,L$.



$= 1000\ cm^3$.

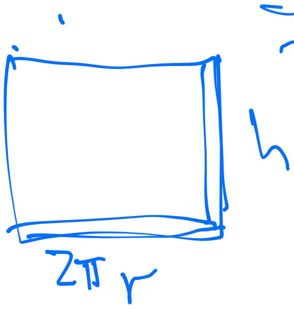


$V = \boxed{\pi r^2 h = 1000\ cm^3}$

top/bottom

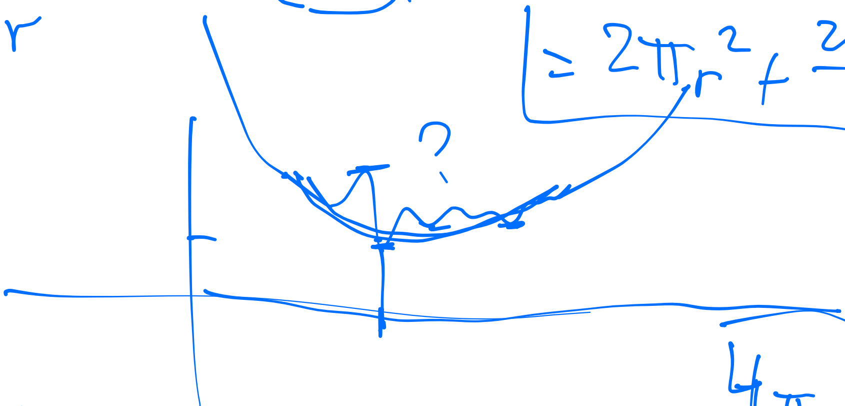
$SA = 2\pi r^2 + 2\pi r h$.

$h = \dfrac{1000}{\pi r^2}$



$SA = 2\pi r^2 + 2\pi r\left(\dfrac{1000}{\pi r^2}\right)$

$\boxed{= 2\pi r^2 + \dfrac{2000}{r}}$



$SA' = 4\pi r - \dfrac{2000}{r^2} = \dfrac{4\pi r^3 - 2000}{r^2} = 0.$

$\boxed{r^* = \sqrt[3]{\dfrac{2000}{4\pi}}\ .\ cm}$ $\boxed{h^* = \dfrac{1000}{\pi}\left(\dfrac{(4\pi)^{2/3}}{2000^{2/3}}\right)\ cm}$

In general, for fixed volume V, ← constant.

$$SA = 2\pi r^2 + \frac{2V}{r} = 2\pi r^2 + 2\pi r h$$

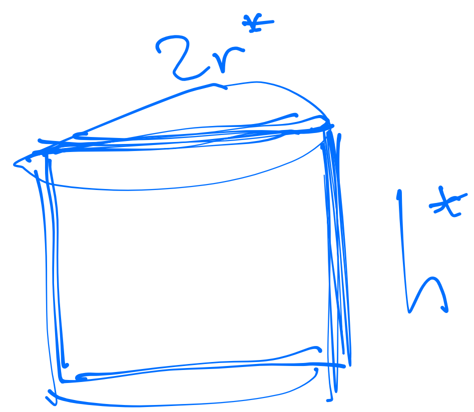
$$SA' = 4\pi r - \frac{2V}{r^2} = \frac{4\pi r^3 - 2V}{r^2} = 0.$$

$$\boxed{r^* = \left(\frac{V}{2\pi}\right)^{1/3}}, \quad V = \pi r^2 h, \quad \boxed{h = \frac{V}{\pi r^2}}.$$

$$h = \frac{V}{\pi}\left(\frac{2^{2/3}\pi^{2/3}}{V^{2/3}}\right) = \boxed{\frac{V^{1/3}}{\pi^{1/3}} \cdot 2^{2/3} = h^*}$$

$$r^* = \frac{V^{1/3}}{\pi^{1/3}} \cdot 2^{-1/3} = \frac{h^*}{2}.$$

$$2r^* = h^*$$

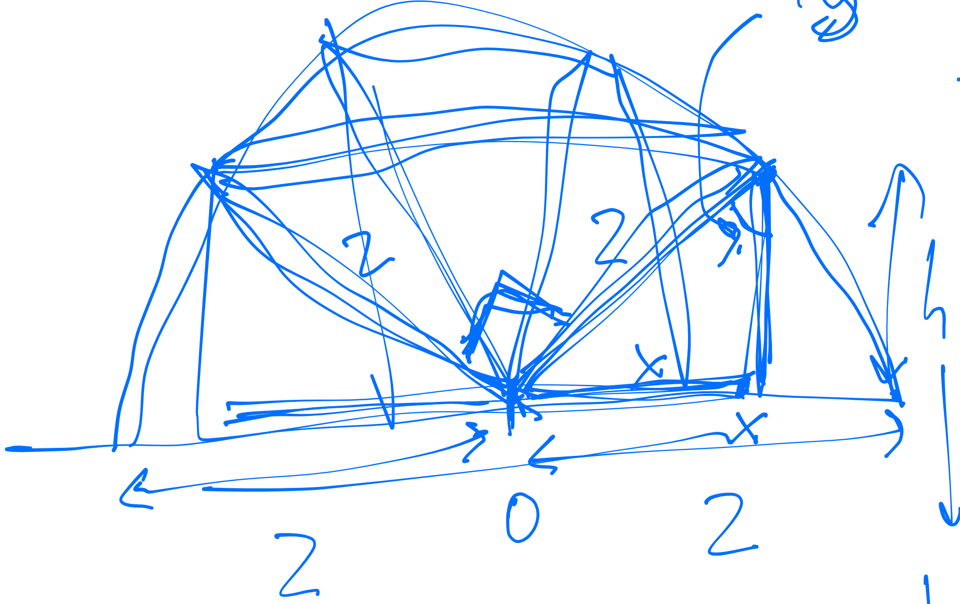


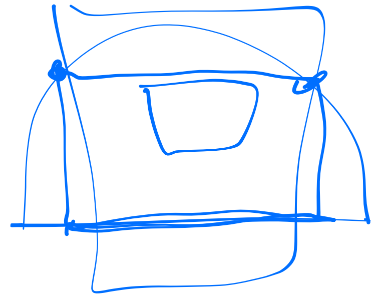
$$1000\ \text{cm}^3 = 1\ \text{L}$$

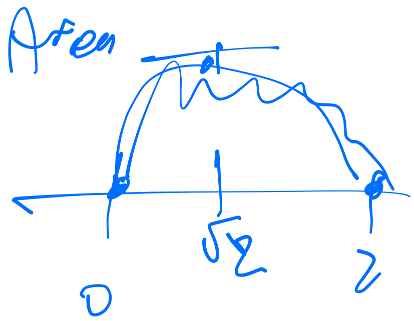
$$1\ \text{m}^3 = 1000\ \text{L}$$

$$(10\ \text{cm})^3 = \left(\frac{1}{10}\ \text{m}\right)^3 = 1\ \text{L}.$$

EXAMPLE 3 A rectangle is to be inscribed in a semicircle of radius 2. What is the largest area the rectangle can have, and what are its dimensions?







base $= 2x$ $\qquad h = \sqrt{4-x^2}$.

Area $= A = 2x\sqrt{4-x^2}$.

$$A' = 2\left[1\cdot\sqrt{4-x^2} + x\,\tfrac{1}{2}(4-x^2)^{-1/2}(-2x)\right].$$

$$= \frac{2}{\sqrt{4-x^2}}\Big[\underbrace{(4-x^2) - x^2}_{4-2x^2}\Big].$$

$$A' = 0 \Rightarrow \boxed{x^* = \cancel{\pm}\sqrt{2}}.$$

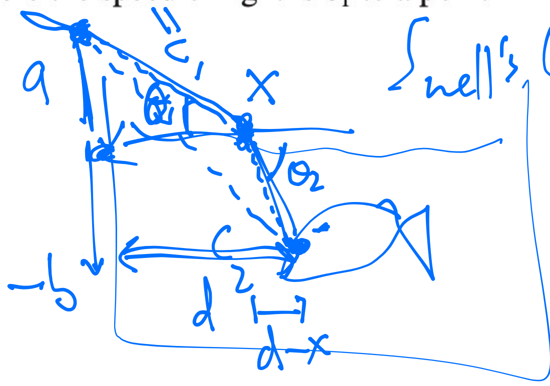
$$\boxed{\begin{array}{l} x^* = \sqrt{2} \\ h^* = \sqrt{2} \end{array}}$$

Optimal dimensions: $2\sqrt{2} \times \sqrt{2}$

area: 4.

EXAMPLE 4 The speed of light depends on the medium through which it travels, and is generally slower in denser media.

Fermat's principle in optics states that light travels from one point to another along a path for which the time of travel is a minimum. Describe the path that a ray of light will follow in going from a point A in a medium where the speed of light is c_1 to a point B in a second medium where its speed is c_2.



Snell's Law

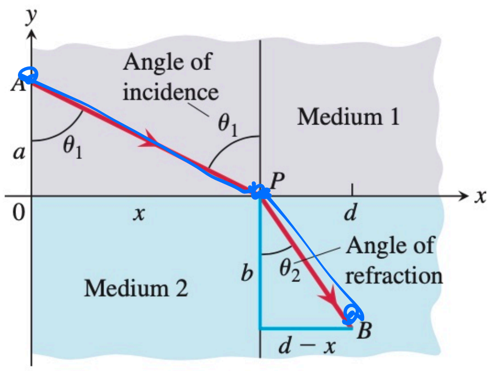


FIGURE 4.41 A light ray refracted (deflected from its path) as it passes from one medium to a denser medium (Example 4).

Want: total time for light to travel from $(0,a)$ to $(x,0)$, to $(d,-b)$.

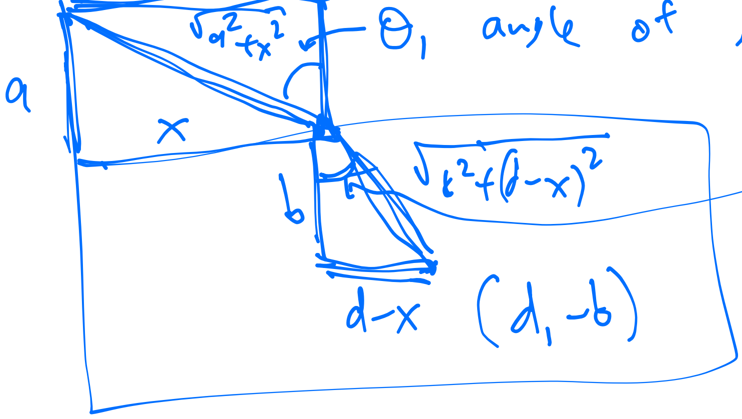
distance$_1$ $= \sqrt{x^2+a^2}$ time$_1$ $= \dfrac{\sqrt{x^2+a^2}}{c_1}$

distance$_2$ $= \sqrt{b^2+(d-x)^2}$ time$_2$ $= \dfrac{\sqrt{b^2+(d-x)^2}}{c_2}$.

Total time: $T = \dfrac{\sqrt{x^2+a^2}}{c_1} + \dfrac{\sqrt{b^2+(d-x)^2}}{c_2}$.

$$T' = \underbrace{\frac{1}{c_1}\cancel{\frac{1}{2}}(x^2+a^2)^{-1/2}(\cancel{2}x)} + \underbrace{\frac{1}{c_2}\cancel{\frac{1}{2}}\left(b^2+(d-x)^2\right)^{-1/2}\left(\cancel{2}(d-x)\right)(-1)} = 0$$

$$\Rightarrow \frac{1}{c_1}\cdot\frac{x}{(x^2+a^2)^{1/2}} = \frac{1}{c_2}\,\frac{d-x}{\sqrt{b^2+(d-x)^2}}$$



θ_1 angle of incidence

θ_2 = angle of refraction.

$$\frac{1}{c_1}\cdot\overbrace{\frac{x}{(x^2+a^2)^{1/2}}}^{\sin\theta_1} = \frac{1}{c_2}\overbrace{\frac{d-x}{\sqrt{b^2+(d-x)^2}}}^{\sin\theta_2}$$

$$\boxed{\text{Snell's law: } \quad \frac{\sin\theta_1}{\sin\theta_2} = \frac{c_1}{c_2}}$$

revenue cost

EXAMPLE 5 Suppose that $r(x) = 9x$ and $c(x) = x^3 - 6x^2 + 15x$, where x represents millions of MP3 players produced. Is there a production level that maximizes profit? If so, what is it?

$$p(x) = r(x) - c(x).$$

$$0 = p'(x) = r'(x) - c'(x).$$

$$p(x) = 9x - x^3 + 6x^2 - 15x$$

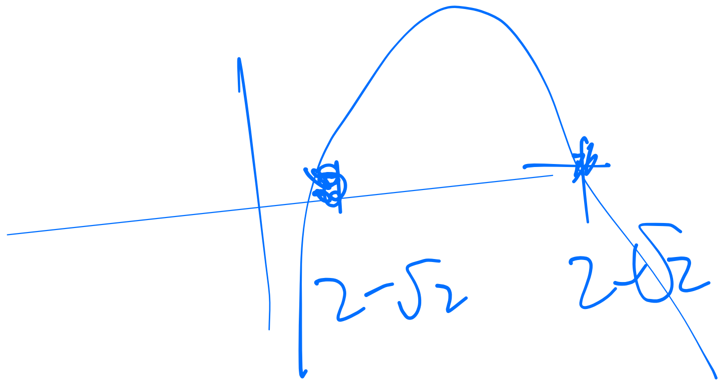
$$p'(x) = -6 + 12x - 3x^2$$

$$0 = -3(x^2 - 4x + 2) = p'(x)$$

$$x = \frac{+4 \pm \sqrt{16 - 4(1)(2)}}{2}$$

$$= 2 \pm \sqrt{2}.$$

p' — $2-\sqrt{2}$ + $2+\sqrt{2}$ —



$2-\sqrt{2}$ = local min $(p''(2-\sqrt{2}) > 0)$

$\boxed{2+\sqrt{2} = \text{local max}}$ $(p''(2+\sqrt{2}) < 0)$.

<u>Recall:</u> F is an <u>antideriv</u> for f on D (domain) if:

$$F'(x) = f(x) \text{ on } D.$$

<u>Thm:</u> If F & G are antideriv's of f, then

$\exists C$ s.t. $F(x) = G(x) + C$

Notation: "indefinite integral".

$\int f(x)\,dx$ ← set of all poss. antiderivs.

Eg: $\int x^5\,dx = \frac{x^6}{6} + C$,

$$\int \frac{1}{1+x^2}\,dx = \tan^{-1}x + C$$

$$\int e^{2x}\,dx = \frac{1}{2}e^{2x} + C$$

$$\int \frac{1}{\sqrt{3-x^2}}\,dx = \frac{1}{\sqrt{3}}\int \frac{1}{\sqrt{1-\left(\frac{x}{\sqrt{3}}\right)^2}}\,dx$$

$$\boxed{\int \frac{1}{\sqrt{1-u^2}}\,du = \sin^{-1} u + C}$$

$$= \sin^{-1}\left(\frac{x}{\sqrt{3}}\right) + C$$