Last time: Curve sketching, local max/min, First Deriv Test, Second Deriv Test, Concave up/down, inflection pts,

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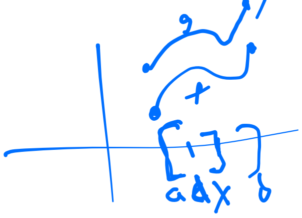
§ Antiderivatives: If $f(x) = 3x^3+2x^2-7x+4$. then $f'(x) = 9x^2+4x-7$. Other direction: If I know $f'(x) = 9x^2+4x-7$, what could f be?.

$f(x) = 3x^3+2x^2-7x+C$, for any constant C.

Thm: If f & g are cont on $[a,b]$, & f', g' exist on (a,b), & $\forall x \in (a,b)$ $f'(x) = g'(x)$ then $\exists C$ s.t. (such that) $\forall x \in [a,b]$, $f(x) = g(x) + C$.



Pf: Let $h(x) = f(x) - g(x)$. (Want to show: $h(x) = C$).

Let $x \in [a,b]$, By MVT on $[a,x]$, $\exists\, d \in (a,x)$ s.t.

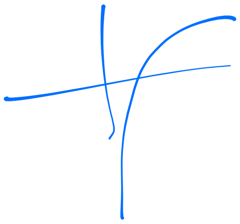
$$\frac{h(x)-h(a)}{x-a} = h'(d) = f'(d) - g'(d) = 0. \Rightarrow h(x) = h(a) =: C.$$

"Differentiation is easy But antidifferentiation is hard".

§ Indeterminate Forms

$\boxed{\frac{0}{0}}$, $\frac{\infty}{\infty}$, 1^∞, $\infty-\infty$, ∞^0, $0\cdot\infty$.

$\ln 0^+ = -\infty$



$0^\infty = 0$.

Thm (Bernoulli's ~~l'Hopital's~~ Rule):

If $\lim_{x\to a} f(x) = 0$ & $\lim_{x\to a} g(\cancel{x}) = 0$ & $\lim_{x\to a} \frac{f'(x)}{g'(x)}$ exists,

Then $$\boxed{\lim_{x\to a}\frac{f(x)}{g(x)} = \lim_{x\to a}\frac{f'(x)}{g'(x)}}$$

Do not confuse with Quotient Rule.

Eg: $\lim_{x\to 0}\frac{x-\sin(3x)}{2x}$,

$f(x) = x - \sin(3x) \xrightarrow[x\to 0]{} 0$

$g(x) = 2x \to 0$, as $x\to 0$.

$\|$

$f'(x) = 1 - \cos(3x)\cdot 3$

$g'(x) = 2$

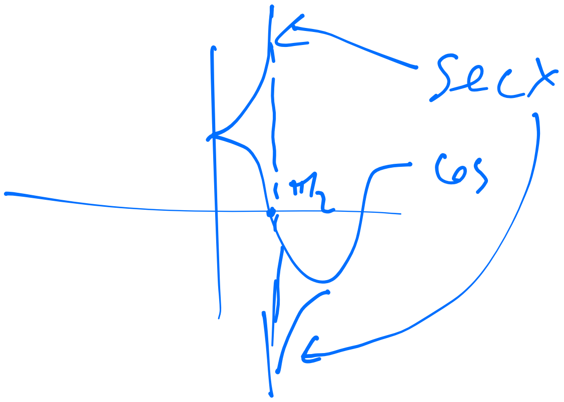
$$\lim_{x\to 0}\frac{1-\overset{\to 1}{\cos(3x)}\cdot 3}{2} = \frac{1-3}{2} = -1.$$

$$\lim_{x\to 0}\frac{1}{2} - \frac{3x}{2x}\cdot\overset{\to 1}{\frac{\sin(3x)}{3x}} = -1.$$

Eg: $$\lim_{x\to 0}\frac{\overset{\to 0}{\sqrt{1+x}-1}}{\underset{\to 0}{x}} \overset{?\checkmark}{\underset{(l'H)}{=}} \lim_{x\to 0}\frac{\frac{1}{2}(1+x)^{-1/2}}{1} = \frac{1}{2}.$$

"OLD way": $\lim_{x\to0}\frac{(\sqrt{1+x}-1)}{x}\frac{(\sqrt{1+x}+1)}{(\sqrt{1+x}+1)} = \lim_{x\to0}\frac{\overset{1}{\cancel{(1+x)-1}}}{\cancel{x}(\sqrt{1+x}+1)} = \underline{\underline{\frac{1}{2}}}.$

Eg: $\lim_{x\to\frac{\pi}{2}}\frac{\sec x \nearrow \pm\infty}{1+\tan x \to \pm\infty} = 1.$



Try separating limits $\lim_{x\to\frac{\pi}{2}^{\pm}}\frac{\cancel{\sec x}\tan x}{\sec^{\cancel{2}}x} = \lim_{x\to\frac{\pi}{2}^{\pm}}\sin x = \underline{1}$

Quiz: $\lim_{x\to\infty}\frac{\ln x \to \infty}{\sqrt{x}\to\infty}$ $\frac{\infty}{\infty}$ Indet. form. Try l'Hopital.

Look at $\lim_{x\to\infty}\frac{\frac{1}{x}}{\frac{1}{2}x^{-1/2}} = \lim_{x\to\infty}\frac{1}{x}\cdot\frac{2x^{1/2}}{1} = \lim_{x\to\infty}\frac{2}{\sqrt{x}} = 0.$

$\Rightarrow \lim_{x\to\infty}\frac{\ln x}{\sqrt{x}} = 0.$

$\frac{\ln e^{1000}}{\sqrt{e^{1000}}} = \frac{1000}{e^{500}} \nearrow$

Eg: $\lim_{x\to0^+}\sqrt{x}\cdot\ln x = \lim_{x\to0^+}\frac{\ln x \to -\infty}{\frac{1}{\sqrt{x}}\to\infty}$

$\sqrt{x}\downarrow\ 0 \cdot (-\infty) \downarrow \ln x$

$\overset{l'H\,✓}{=} \lim_{x\to0^+}\frac{\frac{1}{x}}{-\frac{1}{2}x^{-3/2}} = \lim_{x\to0^+}-2x^{1/2} = 0.$

Eg: $\lim_{x\to 0}\left(\underset{\underset{\pm\infty}{\downarrow}}{\frac{1}{\sin x}} - \underset{\underset{\pm\infty}{\downarrow}}{\frac{1}{x}}\right) = \lim_{x\to 0}\left(\frac{x-\sin x \to 0}{x\cdot\sin x \to 0}\right)$ $\quad \frac{0}{0}$ Indet Form

$\pm\infty - \pm\infty$

$$\overset{?\checkmark}{=} \lim_{x\to 0}\frac{1-\cos x \to 0}{1\cdot\sin x + x\cdot\cos x \to 0,} \overset{?\checkmark}{=} \lim_{x\to 0}\frac{\sin x \to 0}{\cos x + 1\cdot\cos x - x\cdot\sin x \to 2.}$$

$$= 0$$

Eg: $\lim_{x\to\infty}\left(\underbrace{1+\frac{1}{x}}_{\to 1}\right)^{\boxed{x}\to\infty}$ $\qquad \underline{1^{\infty}\text{ ; indet form.}}$

$$= \lim_{x\to\infty}\left(\frac{x+1}{x}\right)^{x} = e \qquad \text{Let} \quad L = \left(1+\frac{1}{x}\right)^{x} = e^{\underline{\ln L}\to 1.} \to e^{1}$$

$$\lim_{x\to\infty}\underline{\underline{\ln L}} = \lim_{x\to\infty}\underset{\underset{\infty}{\searrow}}{x}\cdot\underbrace{\ln\left(1+\frac{1}{x}\right)}_{\underset{0}{\downarrow}} = \lim_{x\to\infty}\frac{\ln\left(1+\frac{1}{x}\right)\to 0}{\frac{1}{x}\to 0}.$$

$\infty \cdot 0$

$$\overset{?}{=} \lim_{x\to\infty}\frac{\frac{1}{1+\frac{1}{x}}\cdot\left(\cancel{\frac{-1}{x^2}}\right)}{\cancel{\frac{-1}{x^2}}} = 1, \Rightarrow \lim_{x\to\infty}\left(1+\frac{1}{x}\right)^{x} = e$$

$$\approx 2,7\ldots$$

Eg: $\lim_{x\to\infty} x^{\boxed{1/x} \to 0} = 1$, $x \searrow \infty$ $\quad \infty^{\boxed{0}}$. Indet Form

$$L = x^{1/x} \to e^0 = 1, \quad \ln L = \frac{1}{x}\ln x \to 0$$

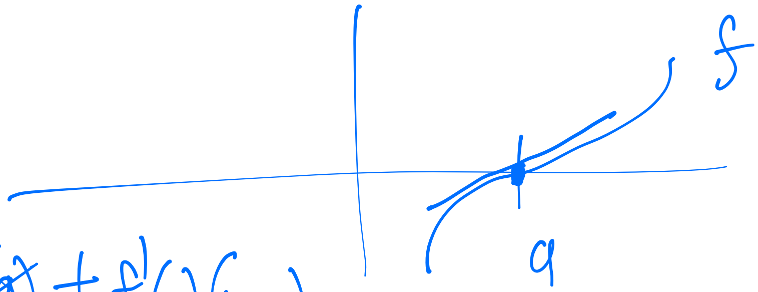
$$\lim_{x\to\infty} \frac{\ln x \to \infty}{x \to \infty} \overset{\checkmark}{=} \lim_{x\to\infty} \frac{\frac{1}{x}}{1} = 0$$

Thm (Bernoulli's "l'Hopital" Rule)

Want $\lim_{x\to a} \frac{f(x)}{g(x)}$ but $f(a) = 0 = g(a)$,

Then if $\lim_{x\to a} \frac{f'(x)}{g'(x)}$ exists, then $\lim_{x\to a} \frac{f(x)}{g(x)} = \lim_{x\to a} \frac{f'(x)}{g'(x)}$.

Pf: $f(a) = 0$



Near a, $\underbrace{f(x)} = \underbrace{\cancel{f(a)} + \underbrace{f'(a)(x-a)}_{\text{differential.}}}$, + small

Same for $g(x) = \overset{0}{\cancel{g(a)}} + g'(a)(x-a) + \text{small}$

$$\Rightarrow \lim_{x \to a} \frac{f(x)}{g(x)} = \lim_{x \to a} \frac{\cancel{f(a)} + f'(a)\cancel{(x-a)} + \cancel{\text{small}}}{\cancel{g(a)} + g'(a)\cancel{(x-a)} + \cancel{\text{small}}} = \lim_{x \to a} \frac{f'(x)}{g'(x)}$$