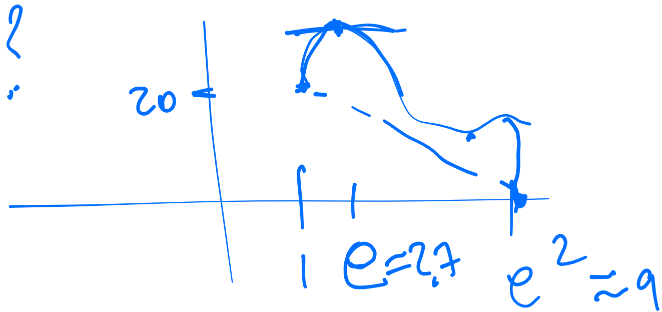
Last time: Differential $dy = f'(x)\,dx$ — small error/perturbation

Linearization, Approximation.

§4. Applications of Derivatives.

Eg: $f(x) = 10x(2-\ln x)$ on $[1, e^2]$.

max, min value on Domain?



$f(1) = 20$

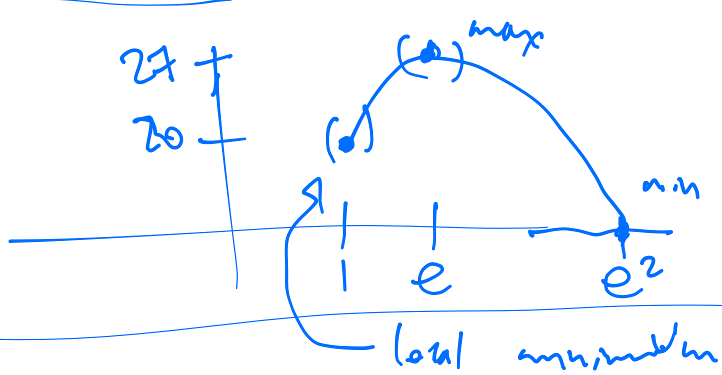
$f(e^2) = 0$

Know: f cont, f' exists.

To "turn around", need: $f'(c)=0$. $f'(x) = 10\{\overbrace{1\cdot(2-\ln x) + x\left(-\frac{1}{x}\right)}^{1-\ln x}\}$

$= 10(1-\ln x) = 0 \Rightarrow x = e$, $\ln x = \ln e = 1$.

Graph:

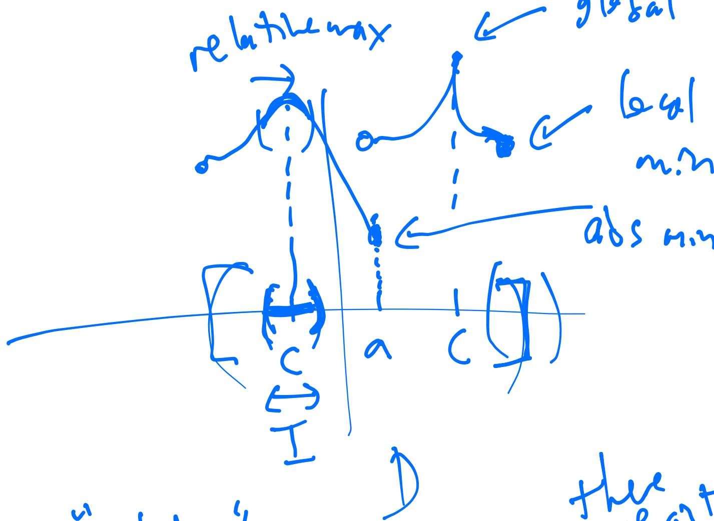


$f(e) = 10e(2-1)$
$= 10e$

Def: Let f be a function on domain D

We say that $c \in D$ is an absolute (global) max

if: $\forall x \in D$, $f(c) \geq f(x)$.



We say that $a \in D$ is an absolute minimum if:

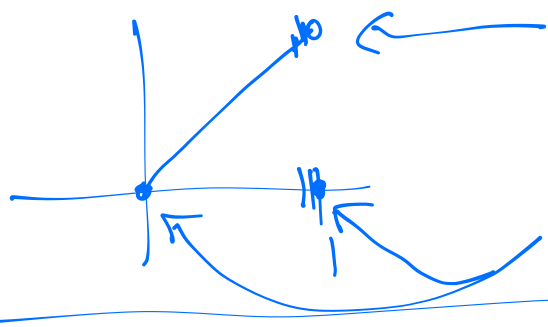
$\forall x \in D$, $f(a) \leq f(x)$

We say that $c \in D$ is a "relative" local max if: $\exists$ (there exists) I open interval, $c \in I$ s.t. (such that) $\forall x \in I \cap D$, $f(c) \geq f(x)$.

min $\leq$

E.g. $f(x) = \begin{cases} x & 0 \leq x < 1 \\ 0 & x = 1 \end{cases}$ $D = [0,1]$.

abs max/min, local max/min. No local/global max.

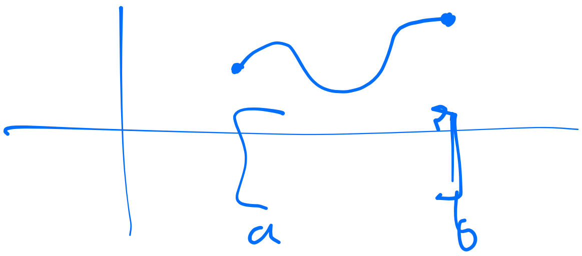


no val c s.t. $f(c) \geq f(x)$, for all x.

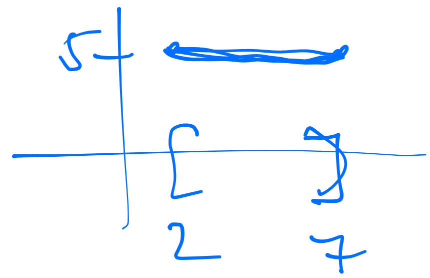
abs min (& local min).

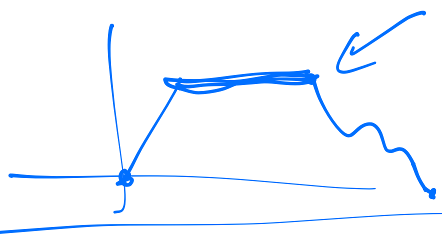
E.g. $f(x) = x$ on $0 \leq x < 1$. No max.

Extreme Value Thm: If f cont on $[a,b]$ closed interval, then f has a abs max & min



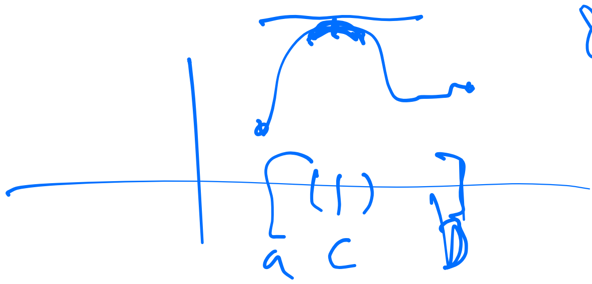
$y = 5$





at abs (& local) max.

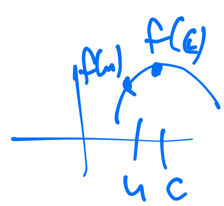
First Deriv Theorem: If f cont on $[a,b]$, & f has local max (min) at interior pt $c \in (a,b)$.



& $f'(c)$ exists. Then $f'(c) = 0$.

Pf: $\lim_{u \to c} \dfrac{f(u) - f(c)}{u - c}$ exists $\Rightarrow \lim_{u \to c^+} = \lim_{u \to c^-}$.

$\forall$ u near c, $f(c) \geq f(u)$



Look at $\displaystyle\lim_{u\to c^-} \frac{f(u)-f(c)}{u-c} = \lim_{u\to c^-} \frac{\le 0}{\le 0} \ge 0$

$\Rightarrow f'(c) \ge 0$. Look at $\displaystyle\lim_{u\to c^+} \frac{\overbrace{f(u)-f(c)}^{\le 0}}{\underbrace{u-c}_{\ge 0}} \le 0$.

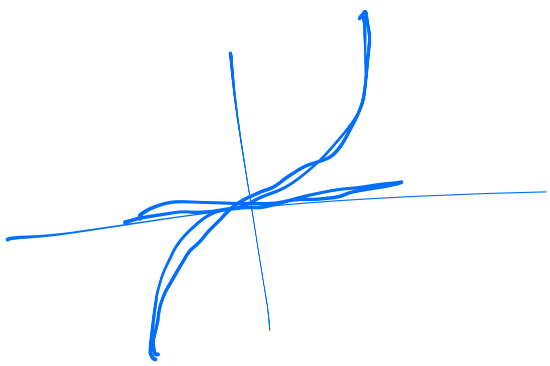
$\Rightarrow f'(c) \le 0$. $\Rightarrow \boxed{f'(c)=0.}$

Def: If $f'(c)=0$ or $f'(c)$ DNE, we say that c is a critical point.

E.g.: $y = x^3$

$y' = 3x^2 = 0$

$\Rightarrow x = 0$



Algorithm/Procedure for finding local/abs max/min.

① Look at end pts ② Critical values, f' DNE or $f'=0$

③ piecewise end pts, both sides

Eval f at these places & compare.

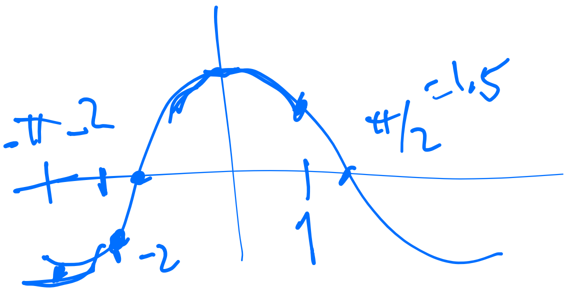
E.g.: $f(x) = \begin{cases} \sin x, & 2 \le x \le 1 \\ \dfrac{1}{x^2}, & 1 < x \le 7 \end{cases}$

Usual suspects: $\boxed{-2,\ 1,\ 7,\ -\frac{\pi}{2}.}$ Crit value: $-\frac{\pi}{2}$

$$f'(x) = \begin{cases} 0 = \cos x, & -2 \le x \le 1 \\ -2x^{-3}, & 1 \le x \le 7 \end{cases}$$

undef at 0 not in domain

no crit values.



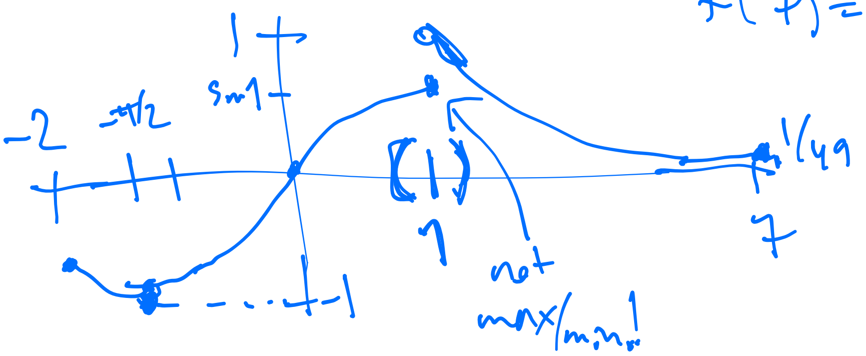
$f(-2) = \sin(-2)$ ← local max

$f(1) = \sin(1)$ ← nothing.

$f(7) = 1/49$, $f(1^+) = 1$ not max

local min

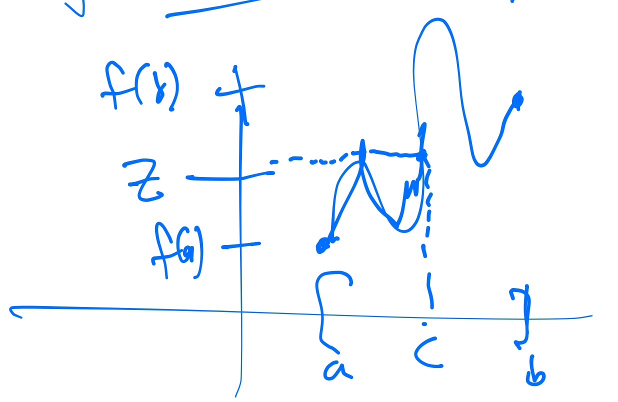
$f\left(\frac{-\pi}{2}\right) = -1.$ abs min



Intermediate Value Thm: f cont on $[a,b]$

Then $\forall z \in [f(a), f(b)]$,

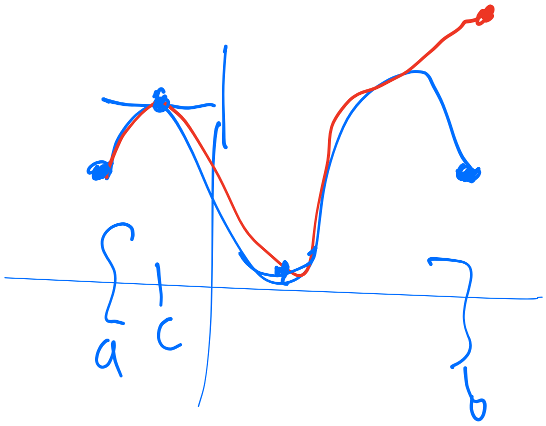
$\exists\ c \in [a,b]$ s.t. $f(c) = z$.



Rolle's Thm: If f cont on $[a,b]$

& f' exists on (a,b) & $f(a) = f(b)$. Then

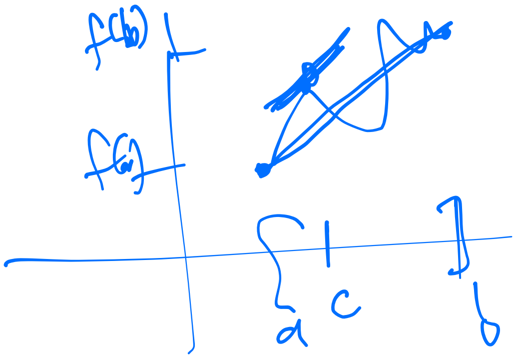
$\exists\ c \in (a,b)$ s.t. $f'(c) = 0$.



Pf: f cont on closed $[a,b]$

By Extreme Val Thm, it has an abs max/min. If max/min at interior pt $c \in (a,b)$, then this is a local max/min. First Deriv Thm $\Rightarrow$ $f'(c)=0$. Alternatively, abs max/min could both be at boundary, $x=a$ & b. But $f(a)=f(b) \longrightarrow f = \text{constant} \Rightarrow f'=0$.

Mean Value Thm: (Ez Puss); Let f be cont on $[a,b]$ & f' exists on (a,b). Then



$\exists\ c \in (a,b)$ s.t.

$$\frac{f(b)-f(a)}{b-a} = f'(c),$$

Pf: let $g(x) = f(x) - f(a) - \dfrac{f(b)-f(a)}{b-a}(x-a)$,

$$g(a) = f(a) - f(a) - \frac{f(b)-f(a)}{b-a}(a-a) = 0.$$

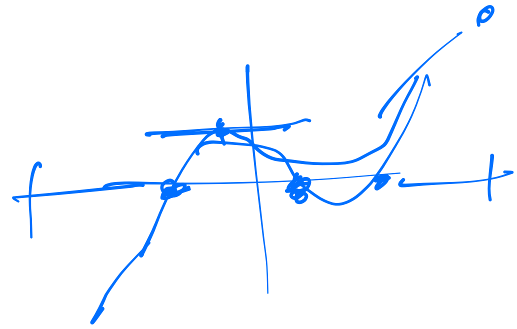
$$g(b) = f(b) - f(a) - \left(\frac{f(b)-f(a)}{\cancel{b-a}}\right)\cancel{(b-a)} = 0.$$

By Rolle, $\exists\, c \in (a,b)$ s.t. $g'(c) = 0$.

$$g'(c) = \boxed{f'(c)} - \boxed{\frac{f(b)-f(a)}{b-a}}(1) = 0.$$

Eg: Prove that x^3+3x+1 has unique root

One root: Intermediate Val Thm.



If $\exists$ another root,

Rolle $\Rightarrow \exists\, f'(c) = 0$.

$f'(x) = 3x^2+3 \neq 0$.