Last time: $\frac{d}{dx}\sin^{-1}x = \frac{1}{\sqrt{1-x^2}}$

$\frac{d}{dx}\tan^{-1}x = \frac{1}{1+x^2}$, $\frac{d}{dx}\ln x = \frac{1}{x}$,

$\frac{d}{dx}f^{-1}(x) = \frac{1}{f'(f^{-1}(x))}$

"Power Rule" $\frac{d}{dx}x^{\alpha}$

"Exponent Rule" $\frac{d}{dx}a^x = a^x \cdot \ln a.$ $(a>0)$

$= \frac{d}{dx}e^{\ln(a^x)} = \frac{d}{dx}e^{x\cdot\ln a}$

$= e^{x\cdot\ln a}\cdot\ln a. = a^x\cdot\ln a.$

"Logarithmic Diff."

Method 1: $y = a^x$.

Take log of both sides:

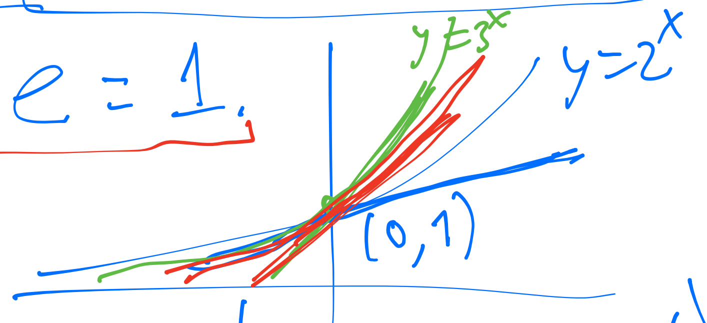
$\ln y = \ln a^x = \underline{\underline{x}}\cdot\underline{\ln a}$

Differentiate both sides:

$\frac{1}{y}\cdot y' = 1\cdot\ln a + \cancel{x\cdot 0}$

$\Rightarrow y' = y\cdot\ln a = a^x\cdot\ln a.$

Obs: When $a=e=2.7\ldots$, $\underbrace{\ln e = 1.}$



Q: Slope of line tangent to $y=2^x$ at $(0,1) = ?$. $y'\Big|_{x=0} = 2^x\cdot\ln 2\Big|_{x=0} = \ln 2 < 1$

Slope of $3^x\Big|_{x=0} = \ln 3 > 1$

For $y=e^x$, slope at $(0,1) = 1$.

Eg: $y = 3^{2x}$, $y' = \underbrace{3^{2x}} \cdot \underbrace{\ln 3 \cdot 2}$. Chain Rule.

$3^{2x} = (3^2)^x = 9^x$, $y' = 9^x \cdot \ln 9 = 3^{2x} \ln 3^2$

$\ln y = 2x \cdot \ln 3$, $\frac{1}{y} \cdot y' = 2 \ln 3$, $\Uparrow$.

Eg: $$y = \frac{(x^2+2)^{3/5} \cdot (x-1)^2}{(x^3+2)^{1/2}(x-5)}$$

$$\ln y = \frac{3}{5}\ln(x^2+2) + 2\ln(x-1) - \frac{1}{2}\ln(x^3+2) - \ln(x-5).$$

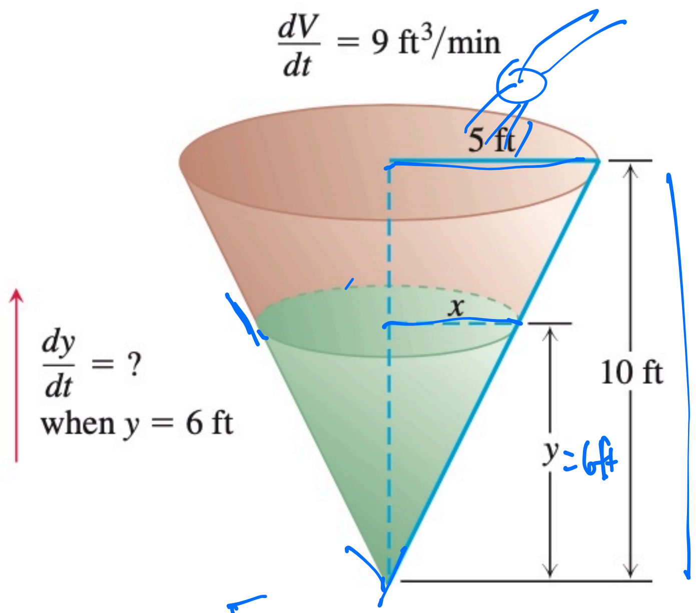
$$\frac{1}{y} \cdot y' = \frac{3}{5}\frac{2x}{x^2+2} + 2 \cdot \frac{1}{x-1} - \frac{1}{2}\frac{3x^2}{x^3+2} - \frac{1}{x-5}.$$

$$y' = \frac{(x^2+2)^{3/5} \cdot (x-1)^2}{(x^3+2)^{1/2}(x-5)}\left[\frac{3}{5}\frac{2x}{x^2+2} + 2 \cdot \frac{1}{x-1} - \frac{1}{2}\frac{3x^2}{x^3+2} - \frac{1}{x-5}\right]$$

$y = x^\alpha$, $\ln y = \alpha \ln x$,

$\frac{1}{y} \cdot y' = \alpha \cdot \frac{1}{x}$, $y' = x^\alpha \cdot \alpha \cdot \frac{1}{x} = \alpha \cdot x^{\alpha - 1}$.

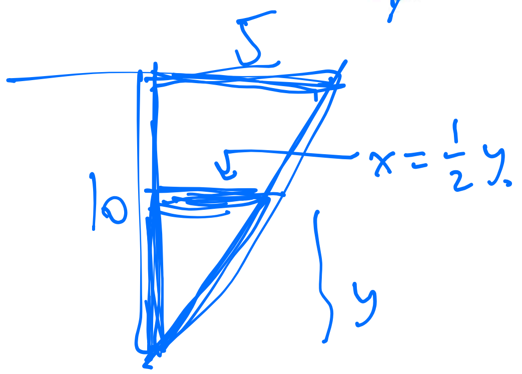
EXAMPLE 1 Water runs into a conical tank at the rate of 9 ft^3/min. The tank stands point down and has a height of 10 ft and a base radius of 5 ft. How fast is the water level rising when the water is 6 ft deep?



$$\boxed{V = \frac{1}{3}\pi x^2 y.}$$

Want: $\frac{dy}{dt}$

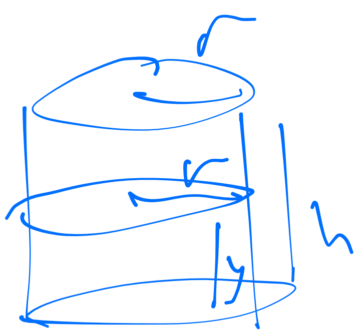
$$\frac{dV}{dt} = 9\ ft^3/m$$



$$V = \frac{1}{3}\pi \left(\frac{y}{2}\right)^2 \cdot y = \frac{\pi}{12} y^3.$$

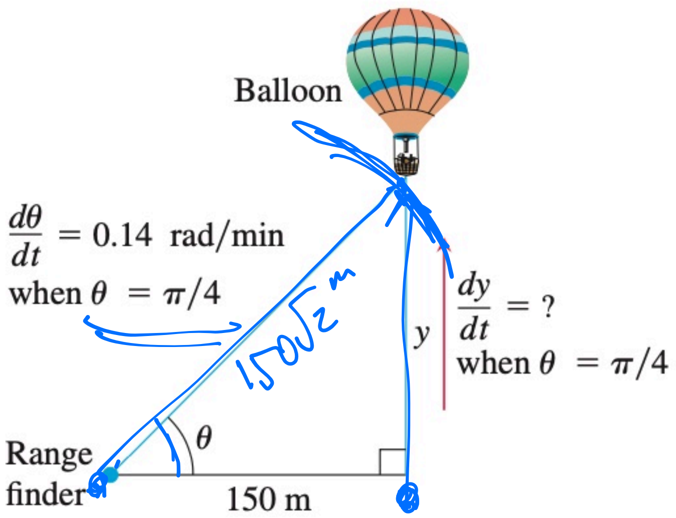
$$9\,\frac{ft^3}{m} = \frac{dV}{dt} = \frac{\pi}{12} \cdot 3y^2 \cdot \frac{dy}{dt} \Big|_{y = 6\,ft} = \frac{\pi}{\cancel{12}} \cdot 3 \cdot \overset{3}{\cancel{36}}\, ft^2 \cdot \frac{dy}{dt}$$

$$\frac{dy}{dt} = 9\,\frac{ft^3}{m} \cdot \frac{1}{9\pi} \frac{1}{ft^2} = \boxed{\frac{1}{\pi}\ \frac{ft}{m}.}$$



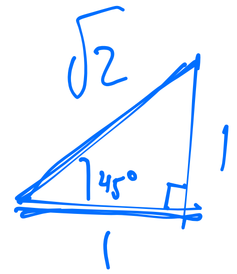
$$V = \pi r^2 \cdot h$$

EXAMPLE 2 A hot air balloon rising straight up from a level field is tracked by a range finder 150 m from the liftoff point. At the moment the range finder's elevation angle is $\pi/4$, the angle is increasing at the rate of 0.14 rad/min. How fast is the balloon rising at that moment?



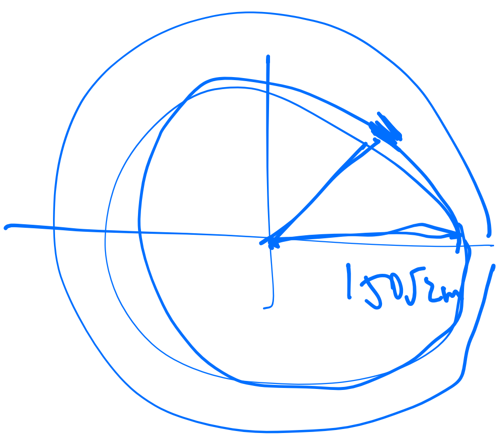
$$\tan\theta = \frac{y}{150 \text{ m}}$$

Take $\frac{d}{dt}$.

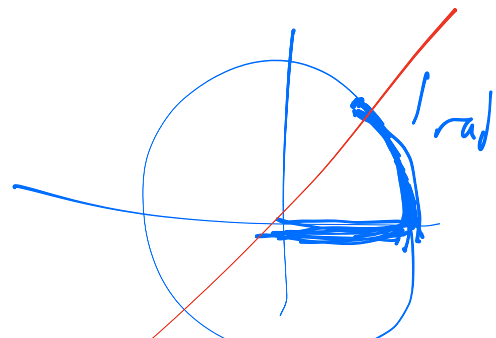


$$\sec^2\theta \cdot \frac{d\theta}{dt} = \frac{1}{150} \cdot \frac{dy}{dt}.$$

$$150 \cdot (\sqrt{2})^2 \cdot 0.14 \frac{\text{rad}}{\text{min}} = \frac{dy}{dt}.$$



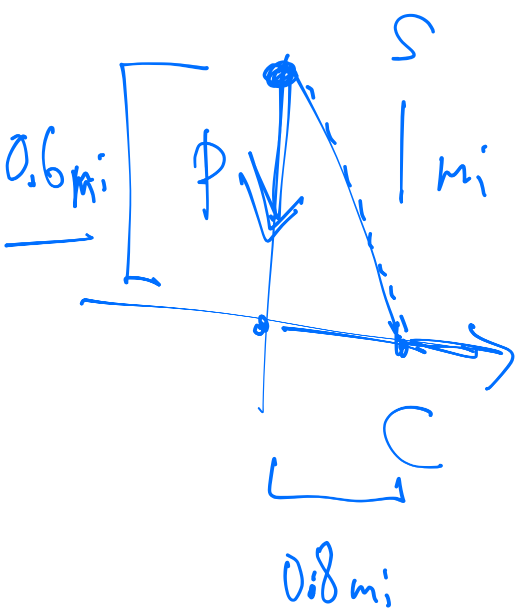
$$\cancel{2\pi \text{ rad} = 2\pi \cdot 150\sqrt{2} \text{ m}}$$



$$\cancel{1 = 150\sqrt{2} \frac{\text{m}}{\text{rad}}}$$

$$\frac{dy}{dt} = 150 \cdot 2 \cdot 0.14 \frac{\cancel{\text{rad}}}{\text{min}} \cdot 150\sqrt{2} \cdot \frac{\text{m}}{\cancel{\text{rad}}} = \boxed{150^2 \cdot 2\sqrt{2} \cdot 0.14 \frac{\text{m}}{\text{min}}}$$

EXAMPLE 3 A police cruiser, approaching a right-angled intersection from the north, is chasing a speeding car that has turned the corner and is now moving straight east. When the cruiser is 0.6 mi north of the intersection and the car is 0.8 mi to the east, the police determine with radar that the distance between them and the car is increasing at 20 mph. If the cruiser is moving at 60 mph at the instant of measurement, what is the speed of the car?



y

x

$$s^2 = x^2 + y^2.$$

$$\frac{ds}{dt} = 20 \text{ mph.}$$

$$\frac{dy}{dt} = -60 \text{ mph.}$$

Want: $\frac{dx}{dt}$.

$$\cancel{2}s\cdot\frac{ds}{dt} = \cancel{2}x\cdot\frac{dx}{dt} + \cancel{2}y\cdot\frac{dy}{dt}.$$

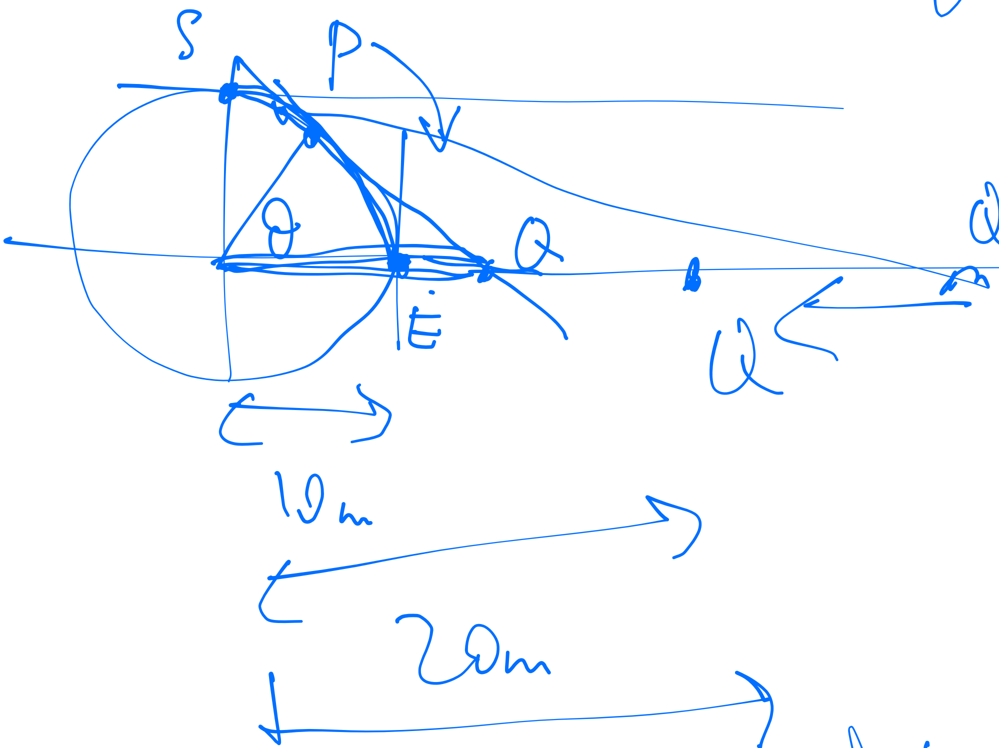
$$1\,mi\cdot 20\,\frac{mi}{hr} = 0.8\,mi\left(\frac{dx}{dt}\right) + 0.6\,mi\left(-60\,\frac{mi}{hr}\right)$$

-36

$$\frac{56\,\frac{mi^2}{hr}}{0.8\,mi} = \frac{dx}{dt} = \boxed{70 \text{ mph}}$$

EXAMPLE 4 A particle P moves clockwise at a constant rate along a circle of radius 10 m centered at the origin. The particle's initial position is $(0, 10)$ on the y-axis, and its final destination is the point $(10, 0)$ on the x-axis. Once the particle is in motion, the tangent line at P intersects the x-axis at a point Q (which moves over time). If it takes the particle 30 sec to travel from start to finish, how fast is the point Q moving along the x-axis when it is 20 m from the center of the circle?

θ from $\pi/2$ to 0

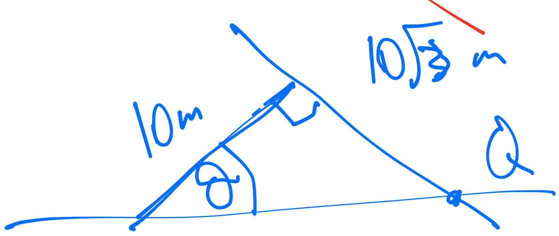


$$\frac{d\theta}{dt} = \frac{-\frac{\pi}{2}\text{ rad}}{30\text{ sec.}}$$

$$\boxed{\frac{d\theta}{dt} = \frac{\pi}{60} \cancel{\cdot \frac{10\text{ m}}{\text{sec.}}}}$$

X = dist Q to $(0,0)$.

$$\cancel{\frac{-\pi}{6}\frac{\text{m}}{\text{sec}} \cdot 30\text{ secs}} = 10 \cdot \frac{\pi}{2}\text{ m}$$



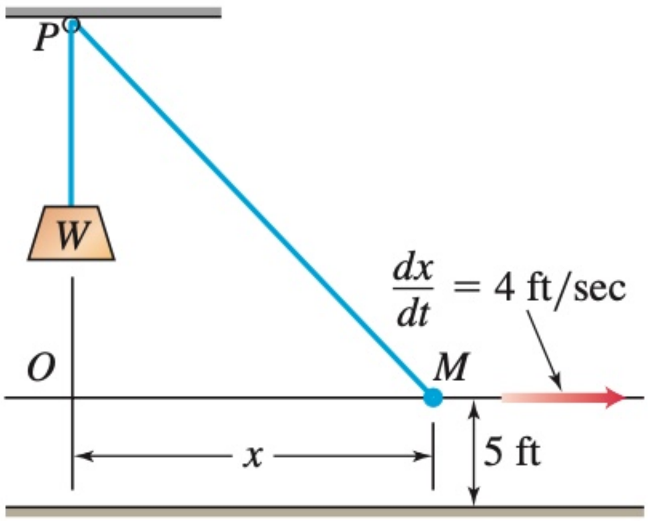
$X = 20$

$$\boxed{\sec\theta = \frac{x}{10\text{ m}}.}$$

$$\underbrace{\sec\theta}_{2} \cdot \underbrace{\tan\theta}_{\sqrt{3}} \cdot \underbrace{\frac{d\theta}{dt}}_{\left(-\frac{\pi}{6}\right)} = \frac{1}{10\text{ m}} \frac{dx}{dt}, \quad \frac{dx}{dt} = \frac{-10 \cdot 2\sqrt{3}\,\pi}{6\text{ s}}\ \text{m}^2$$

EXAMPLE 5 A jet airliner is flying at a constant altitude of 12,000 ft above sea level as it approaches a Pacific island. The aircraft comes within the direct line of sight of a radar station located on the island, and the radar indicates the initial angle between sea level and its line of sight to the aircraft is 30°. How fast (in miles per hour) is the aircraft approaching the island when first detected by the radar instrument if it is turning upward (counterclockwise) at the rate of $2/3$ deg/sec in order to keep the aircraft within its direct line of sight?

EXAMPLE 6 Figure 3.38a shows a rope running through a pulley at P and bearing a weight W at one end. The other end is held 5 ft above the ground in the hand M of a worker. Suppose the pulley is 25 ft above ground, the rope is 45 ft long, and the worker is walking rapidly away from the vertical line PW at the rate of 4 ft/sec. How fast is the weight being raised when the worker's hand is 21 ft away from PW?



(a)