October 2014 (revised Nov 2014)

Letter to Jayadev Athreya about Ford Circles
from Alex Kontorovich

Dear Jayadev,

I enjoyed reading your paper with Cobeli and Zaharescu on the "Radial Density" in Apollonian packings (`arXiv:1409.6352`). A key reduction of your main theorem (Theorem 1.1) is to the case of Ford circles; here is a reproduction of your Figure 5:

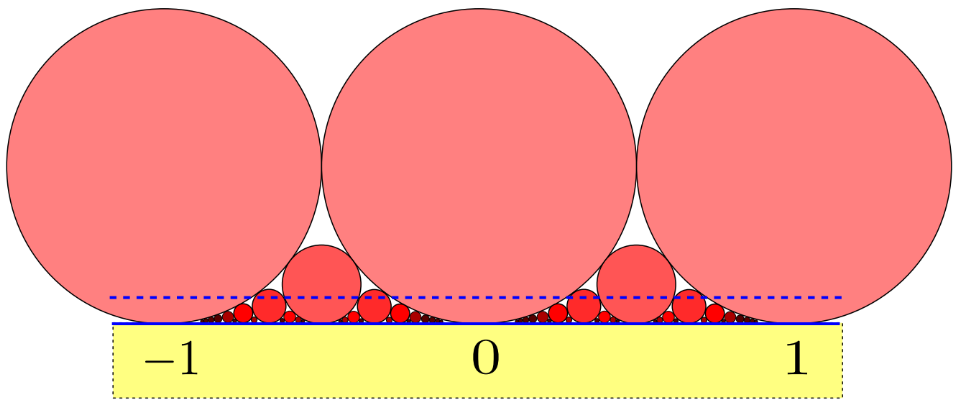

FIGURE 5. An example of a periodic packing is the Farey-Ford Packing $\mathcal{F}$. Here, if we take C_0 to be the segment $[0, 1]$, $\mathcal{F}_0$ consists of Ford Circles, based at $p/q \in [0, 1]$, diameter $1/q^2$, and C_ϵ is the segment $[0, 1] + \epsilon i$.

The question is about the proportion of this line at height $\epsilon > 0$ spending inside the circles, as a function of $\epsilon \to 0$. In section 2, you let the variable h play the role of ϵ, call this quantity $L(h)$, and derive after a simple geometric calculation that

$$L(h) = \sum_{q \le 1/\sqrt{h}} \sum_{(a,q)=1} 2\sqrt{\left(\frac{1}{2q^2}\right)^2 - \left(h - \frac{1}{2q^2}\right)^2}. \tag{1}$$

Your main theorem on this (Theorem 2.1) states that $L(h) = 3/\pi + O(\sqrt{h}|\log h|)$ as $h \to 0$, and you show beautiful and intriguing oscillatory plots illustrating this convergence in Figures 6 and 7, reproduced below. (The corners occur at $h = 1/n^2$ for integers n.)

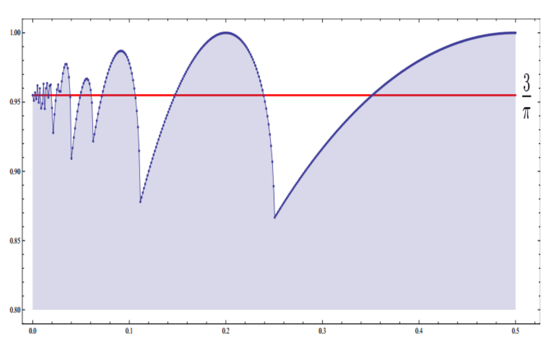

FIGURE 6. The graph of $L(h)$, $0 < h \le 0.57$.

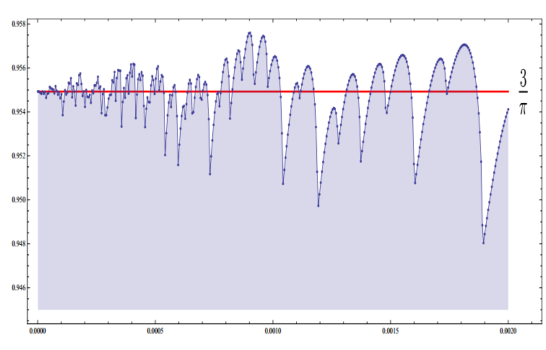

FIGURE 7. The graph of $L(h)$, $0 < h \le 0.002$.

The purpose of my letter is to explain the oscillations and precise nature of these pictures. In fact, the explanation below is a standard application of the theory of automorphic forms, so I would be surprised if this is the first time it's observed. Either way, the claim is that $L(h)$ can be expressed explicitly in terms of the Riemann zeta function. Here is the statement.

Claim: Define $\varphi(s)$ by

$$\varphi(s) := \sqrt{\pi}\frac{\Gamma(s-\frac{1}{2})}{\Gamma(s)}\frac{\zeta(2s-1)}{\zeta(2s)}.$$

Then

$$L(h) = \frac{3}{\pi} + \frac{\sqrt{h}}{2\pi}\int_{\mathbb{R}} \left(h^{it} + \varphi(\tfrac{1}{2}+it)h^{-it}\right)\frac{dt}{\frac{1}{2}+it}. \tag{2}$$

The proof is very simple. Mimicking your section 2.2 (with a slight tweak), let

$$f(z) := \sum_{\gamma\in\Gamma_\infty\backslash\Gamma} \mathbf{1}_{\{\mathfrak{Im}(\gamma z)\geq 1\}}, \tag{3}$$

where $\Gamma = \mathrm{SL}(2,\mathbb{Z})$ and $\Gamma_\infty = \left(\begin{smallmatrix}1 & \mathbb{Z}\\ & 1\end{smallmatrix}\right)$. Then

$$L(h) = \int_0^1 f(x+ih)dx.$$

By the spectral decomposition of automorphic forms (see [IK04, Thm 15.5]), we have:

$$f(z) = \frac{\langle f,1\rangle}{\langle 1,1\rangle} + \sum_j \langle f,\varphi_j\rangle\,\varphi_j(z) + \frac{1}{4\pi}\int_{\mathbb{R}}\left\langle f, E(\tfrac{1}{2}+it,*)\right\rangle E(\tfrac{1}{2}+it,z)dt,$$

where φ_j is an orthonormal basis of Maass cusp forms and

$$E(s,z) = \sum_{\gamma\in\Gamma_\infty\backslash\Gamma} \mathfrak{Im}(\gamma z)^s \tag{4}$$

is the Eisenstein series (analytically continued beyond its original region of convergence). What we really want is not f but its integral over a horocycle of height h; thus

$$\begin{aligned} L(h) &= \int_0^1 f(x+ih)dx \\ &= \frac{\langle f,1\rangle}{\langle 1,1\rangle} + \sum_j \langle f,\varphi_j\rangle \int_0^1 \varphi_j(x+ih)dx \\ &\qquad + \frac{1}{4\pi}\int_{\mathbb{R}}\left\langle f, E(\tfrac{1}{2}+it,*)\right\rangle \int_0^1 E(\tfrac{1}{2}+it,x+ih)dx\,dt. \end{aligned}$$

(The interchange of orders must be justified.)

Now simplify terms. Because the φ_j are cusp forms, their contribution vanishes. We have $\langle f,1\rangle = 1$ and $\langle 1,1\rangle = \mathrm{vol} = \pi/3$; thus the main term is determined. Next we simplify the last term. This is just the constant Fourier coefficient of the Eisenstein series, which is (see [IK04, (15.13)]):

$$\int_0^1 E(s,x+ih)dx = h^s + \varphi(s)h^{1-s}.$$

Finally, unfolding the inner product gives

$$\langle f, E(s,*)\rangle = \int_{\Gamma_\infty\backslash\mathbb{H}} \mathbf{1}_{\{\Im\mathfrak{m} z\geq 1\}}\overline{E(s,z)}dz = \int_1^\infty \int_0^1 \overline{E(s,x+iy)}dx\frac{dy}{y^2}.$$

Again using the Fourier expansion of the Eisenstein series, we obtain

$$\langle f, E(s,*)\rangle = \int_1^\infty (\overline{y^s+\varphi(s)y^{1-s}})\frac{dy}{y^2} = \frac{1}{1-\bar{s}} + \varphi(\bar{s})\frac{1}{\bar{s}}.$$

Putting everything together, we obtain

$$L(h) = \frac{3}{\pi} + \frac{1}{4\pi}\int_{\mathbb{R}} \left(\frac{1}{\frac{1}{2}+it} + \varphi(\tfrac{1}{2}-it)\frac{1}{\frac{1}{2}-it}\right)\left(h^{\frac{1}{2}+it} + \varphi(\tfrac{1}{2}+it)h^{\frac{1}{2}-it}\right)dt.$$

The claim then follows using $|\varphi| = 1$ on the $\mathfrak{Re}(s) = 1/2$ line, and observing the symmetry $t \mapsto -t$.

By the Prime Number Theorem, one can then prove that $L(h) = 3/\pi + o(\sqrt{h})$. Alternatively, one can observe that $f(z)$ is itself an Eisenstein-like series, whence by taking a Mellin transform/inverse, and shifting contours further, one sees (as in Zagier [Zag81] and Sarnak [Sar81]) that

$$L(h) = 3/\pi + O(h^{3/4-\varepsilon}) \tag{5}$$

if and only if the Riemann Hypothesis holds. To make this more precise, let $\chi(y) := \mathbf{1}_{y\geq 1}$ and observe that for $\mathfrak{Re}(s) > 0$, we have the "Mellin" transform/inverse pair:

$$\tilde{\chi}(s) \ := \ \int_0^\infty \chi(y)y^{-s}\frac{dy}{y} = \frac{1}{s}, \qquad \chi(y) \ = \ \frac{1}{2\pi i}\int_{(2)} \tilde{\chi}(s)y^s ds.$$

Then comparing (3) and (4) with the above shows that:

$$f(z) = \sum_{\gamma\in\Gamma_\infty\backslash\Gamma} \chi(\Im\mathfrak{m}(\gamma z)) = \sum_{\gamma\in\Gamma_\infty\backslash\Gamma} \frac{1}{2\pi i}\int_{(2)} \tilde{\chi}(s)\Im\mathfrak{m}(\gamma z)^s ds = \frac{1}{2\pi i}\int_{(2)} E(z,s)\frac{ds}{s}.$$

Then

$$L(h) = \int_0^1 f(x+ih)dx = \frac{1}{2\pi i}\int_{(2)}\int_0^1 E(x+ih,s)dx\frac{ds}{s} = \frac{1}{2\pi i}\int_{(2)} \left(h^s + \varphi(s)h^{1-s}\right)\frac{ds}{s}.$$

Pulling contours from the $\mathfrak{Re}(s) = 2$ line to the $\mathfrak{Re}(s) = 1/2$ line (and recovering the pole at $s = 1$) gives (2) again, and pulling further to the $\mathfrak{Re}(s) = 1/4 + \varepsilon$ line (on RH) gives (5). (Of course once you use Shah's equidistribution theorem to go from Ford circles to general "radial densities" in other Apollonian packings, this rate is lost.)

Best wishes,

Alex

References

[IK04] Henryk Iwaniec and Emmanuel Kowalski. *Analytic number theory*, volume 53 of *American Mathematical Society Colloquium Publications*. American Mathematical Society, Providence, RI, 2004.

[Sar81] Peter Sarnak. Asymptotic behavior of periodic orbits of the horocycle flow and Eisenstein series. *Comm. Pure Appl. Math.*, 34(6):719–739, 1981.

[Zag81] D. Zagier. Eisenstein series and the Riemann zeta function. In *Automorphic forms, representation theory and arithmetic (Bombay, 1979)*, volume 10 of *Tata Inst. Fund. Res. Studies in Math.*, pages 275–301. Tata Inst. Fundamental Res., Bombay, 1981.