A TAXONOMY OF CRYSTALLOGRAPHIC SPHERE PACKINGS

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Abstract. The Apollonian circle packing, generated from three mutually-tangent circles in the plane, has inspired over the past half-century the study of other classes of space-filling packings, both in two and in higher dimensions. Recently, Kontorovich and Nakamura introduced the notion of crystallographic sphere packings, n-dimensional packings of spheres with symmetry groups that are isometries of $\mathbb{H}^{n+1}$, which can be related to configurations of planes in $\mathbb{H}^{n+1}$ for various quadratic forms in $n + 2$ variables. When applied in conjunction with the Koebe-Andreev-Thurston Theorem, Kontorovich and Nakamura's Structure Theorem guarantees crystallographic packings to be generated from polyhedra in $n = 2$. The Structure Theorem similarly allows us to generate packings from the reflective extended Bianchi groups in $n = 2$ by applying Vinberg's algorithm to obtain the appropriate Coxeter diagrams. In $n > 2$, the Structure Theorem when used with Vinberg's algorithm allows us to explore whether certain Coxeter diagrams in $\mathbb{H}^{n+1}$ for a given quadratic form admit a packing at all. Kontorovich and Nakamura's Finiteness Theorem shows that there exist only finitely many classes of these packings, all of which exist in dimensions $n < 21$. In this work, we systematically determine and enumerate crystallographic sphere packings arising from polyhedra on up to seven vertices, all crystallographic packings arising from Bianchi groups, and all known examples of crystallographic packings arising from higher dimensional quadratic forms.

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1. Introduction

Definition 1 (sphere packing). A *sphere packing* in $\mathbb{R}^n \cup \{\infty\}$ is a collection of spheres that:

- are oriented to have mutually disjoint interiors, and
- densely fill up space, so that every point in $\mathbb{R}^n$ lies on the interior or the boundary of a sphere in the packing.

Definition 2 (crystallographic sphere packing). A *crystallographic sphere packing* in $\mathbb{R}^n$ is a sphere packing generated by a finitely generated reflection group $\Gamma < \text{Isom}(\mathbb{H}^{n+1})$ [KN17].

Definition 3. Consider a finite collection of oriented n-spheres $\widetilde{\mathcal{C}} = \mathcal{C} \sqcup \widehat{\mathcal{C}}$ such that a sphere packing is generated by reflections through $\widehat{\mathcal{C}}$ acting on $\mathcal{C}$. Then $\widetilde{\mathcal{C}}$ is known as a *supercluster*, $\mathcal{C}$ is called a *cluster*, and $\widehat{\mathcal{C}}$ is called a *cocluster*.

The Structure Theorem from [KN17] allows us to identify crystallographic sphere packings as finite collections of generating spheres.

Theorem 4 (Structure Theorem for Crystallographic Packings). *If $\widetilde{\mathcal{C}}$ can be decomposed into $\mathcal{C} \sqcup \widehat{\mathcal{C}}$ such that*

- *any two spheres in $\mathcal{C}$ are disjoint or tangent, and*
- *every sphere in $\widehat{\mathcal{C}}$ is disjoint, tangent, or orthogonal to any sphere in $\mathcal{C}$,*

then Γ, a group generated by reflections through $\widehat{\mathcal{C}}$, produces a crystallographic sphere packing via $\Gamma \cdot \mathcal{C}$.

Definition 5 (superpacking). A *superpacking* is a configuration of spheres generated by the action $\widetilde{\Gamma} \cdot \mathcal{C}$, where $\widetilde{\Gamma}$ is generated by reflections in both $\widehat{\mathcal{C}}$ and $\mathcal{C}$.

Note that this is not a packing in the sense of Definition 1 because the interiors are not necessarily mutually disjoint [KN17].

The following theorem from [KN17] motivates our work towards classifying all crystallographic sphere packings.

Theorem 6 (Finiteness Theorem). *Up to commensurability of $\widetilde{\Gamma}$, there are finitely many crystallographic sphere packings, all of which exist in dimension $n < 21$.*

Further, all crystallographic packings arise from one of the three sources explored in this work.

2. FURTHER OBJECTS

Definition 7 (circle inversion/reflection). To invert about $\partial B_z(r)$ (the points in $\widehat{\mathbb{R}^n}$ at distance exactly r from z), send the point $x \in \widehat{\mathbb{R}^n}$ at distance $d = |x - z|$ from z to the point on the ray through x beginning at z at distance $\frac{r^2}{d}$. (This also swaps z and ∞.)

A symmetric $(n+2) \times (n+2)$ matrix Q with signature $(1, n+1)$ gives rise to a model of hyperbolic space through $\{x \in \mathbb{R}^{n+2} \mid \langle x, x \rangle_Q = r\}$ for some $r \in \mathbb{R}^-$ and $\langle x, y \rangle_Q = xQy^T$. We use here

$$Q_n = \begin{pmatrix} & \frac{1}{2} & \\ \frac{1}{2} & & \\ & & -I_n \end{pmatrix}, \tag{8}$$

where the subscript n may be omitted depending on context.

Lemma 9. *When viewed in an upper half-space model, $\mathbb{H}^{n+1}$'s planes are precisely the hemihyperspheres with circumferences on the boundary of space, namely $\mathbb{R}^n$. In the case of a hyperplane as the boundary, the plane is a plane in the Euclidean sense which is orthogonal to the boundary hyperplane.*

Definition 10 (oriented spheres). For $r \in \mathbb{R}\backslash\{0\}$, the sphere centered at z with radius r is the set $\partial B_z(r)$, and we define its interior to be $\{x \in \widehat{R^n} \mid (r - |z - x|)\operatorname{sign} r > 0\}$.

Definition 11 (inversive coordinates). An oriented sphere centered at z with radius $r \in \mathbb{R}\backslash\{0\}$ may be represented by *inversive coordinates* consisting of $\left(\widehat{b}, b, bz\right)$ for

$$b = \frac{1}{r} \text{ and } \widehat{b} = \frac{1}{\widehat{r}},$$

where $\widehat{r}$ is the oriented radius of $\partial B_z(r)$ reflected through $\partial B_0(1)$. We refer to b as the *bend* and $\widehat{b}$ as the *co-bend*.

As shown in [Kon17], any n-dimensional inversive coordinate v satisfies $\langle v, v \rangle_Q = -1$. This leads to the following definition for reflecting about an oriented sphere:

Definition 12 (reflection matrix). The *reflection matrix* about $\widehat{v}$ is given by $R_{\widehat{v}} = I_n + 2Q\widehat{v}^T\widehat{v}$.

This arises from the formula for reflection of v about $\widehat{v}$ with the inner product $\langle \cdot, \cdot \rangle$ given as $R_{\widehat{v}}(v) = v - 2 \frac{\langle v, \widehat{v} \rangle}{\langle \widehat{v}, \widehat{v} \rangle} \widehat{v}$ which expands in our inner product as $R_{\widehat{v}}(v) = v + 2vQ\widehat{v}^T\widehat{v}$ and is a right-acting matrix on v.

We are also equipped to use inversive coordinates to represent "degenerate spheres" of "radius infinity," i.e. codimension-1 hyperplanes in $\mathbb{R}^n$.

Lemma 13. *Consider a hyperplane H with $\operatorname{codim} H = 1$, normal vector $\widehat{n}$, and $P \in H$ the closest point to the origin, and let $S_r \subset \mathbb{R}^n$ be the sphere of radius r tangent to P with interior on opposite half-planes from the origin. Then,*

$$\lim_{r \to \infty} bz = \widehat{n}, \tag{14}$$

$$\lim_{r \to \infty} \widehat{b} = 2\,|P| \tag{15}$$

for $z, b, \widehat{b}$ dependent on r.

Proof. To prove (14): $\widehat{n} = P/\,|P|$ and $z = P + \widehat{n}r = P + Pr/\,|P| = P(1 + r/\,|P|)$ and hence $bz = P/r + \widehat{n}$; $\lim\limits_{r \to \infty} bz = \lim\limits_{r \to \infty} P/r + \widehat{n} = \widehat{n}$.

To prove (15):

$$
\begin{aligned}
\lim_{r \to \infty} \widehat{b} &= \lim_{r \to \infty} \frac{|z|^2 - r^2}{r} \\
&= \lim_{r \to \infty} \left(\frac{|z|^2}{r} - r \right) \\
&= \lim_{r \to \infty} \left(\frac{|P|^2 (1 + r/\,|P|)^2}{r} - r \right) \\
&= \lim_{r \to \infty} \left(\frac{(|P| + r)^2}{r} - r \right) \\
&= \lim_{r \to \infty} \left(\frac{|P|^2}{r} + 2\,|P| \right) \\
&= 2\,|P|\,.
\end{aligned}
$$

$\square$

This enables us to legitimately view hyperplanes as the limits of increasingly large spheres.

Definition 16 (Coxeter diagram)**.** A *Coxeter diagram* encodes the walls of a Coxeter polyhedron (whose dihedral angles are all of the form $\frac{\pi}{n}$) as nodes in a graph, where we draw between nodes corresponding to walls meeting at dihedral angle θ if they

meet at all

$$\begin{cases} \text{a thick line,} & \text{if walls are tangent at a point (including } \infty). \\ \text{no line,} & \text{if } \theta = \frac{\pi}{2}. \\ \text{a dashed line,} & \text{if walls are disjoint.} \\ n - 2 \text{ lines,} & \text{if } \theta = \frac{\pi}{n}. \end{cases}$$

By Theorem 4, a Coxeter diagram can be used to visually identify clusters by identifying those vertices that are adjacent to all other vertices exclusively by thick, dashed, or no lines.

Definition 17 (Gram matrix). If V is a rank-$(n+2)$ matrix of inversive coordinates, then its *Gram matrix* is defined as VQV^T.

The rows and columns of a Gram matrix correspond to walls of a Coxeter polyhedron, where the entries are determined by

$$G_{i,j} = \langle v_i, v_j \rangle_Q = \begin{cases} -1, & v_i = v_j. \\ 1, & v_i || v_j. \\ 0, & v_i \perp v_j. \\ \cos(\theta), & \theta_{v_i, v_j}. \\ \cosh(d), & d = \text{hyperbolic distance}(v_i, v_j). \end{cases}$$

A Gram matrix encodes the same information as a Coxeter diagram, but also includes the hyperbolic distance between two disjoint walls.

Definition 18 (bend matrix). For V, a rank-$(n+2)$ collection of inversive coordiantes, and R, the reflection matrix about a n-sphere, a left-acting *bend matrix* B satisfies the equation

$$BV = VR. \tag{19}$$

Bend matrices can be used to compute the inversive coordinates of a packing. They are a useful tool in proving integrality of packings, as will be demonstrated in §3.

There are three sources that can be used to generate crystallographic packings, whose details will be elaborated upon in the coming sections. The following diagram provides a rough outline for how these packings can be obtained:

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┌──────────────┐        ┌──────────────────┐      ┌────────────────┐
│   Polyhedra  │        │ Dimension n ⩾ 3* │      │  Bianchi groups│
└──────────────┘        └──────────────────┘      └────────────────┘
        │                        │                         │
        ▼                        ▼                         │
  ╱─────────────────╲     ┌──────────────────┐             │
 ╱ Apply K-A-T Theorem╲   │Select quadratic form│◄─────────┘
╱───────────────────╲─╱   └──────────────────┘
        │                        │
        │                        ▼
        │              ┌──────────────────────────┐
        │              │Apply Vinberg's algorithm [Vin72]│        [BM13; Mcl10; Vin72]
        │              └──────────────────────────┘
        │                        │
        │                        ▼
        │              ┌──────────────────────────┐
        │              │Obtain fundamental polyhedron│
        │              └──────────────────────────┘
        │                        │
        │                        ▼
        │              ┌──────────────────────────┐
        │              │Describe with Coxeter diagram│◄──────────┘
        │              └──────────────────────────┘
        │                        │
        ▼                        ▼
  ╱──────────────────────────────────╲
 ╱      Apply Structure Theorem        ╲
╱───────────────────────────────────────╲
                 │
                 ▼
      ┌──────────────────────┐
      │ Generate circle packing│
      └──────────────────────┘
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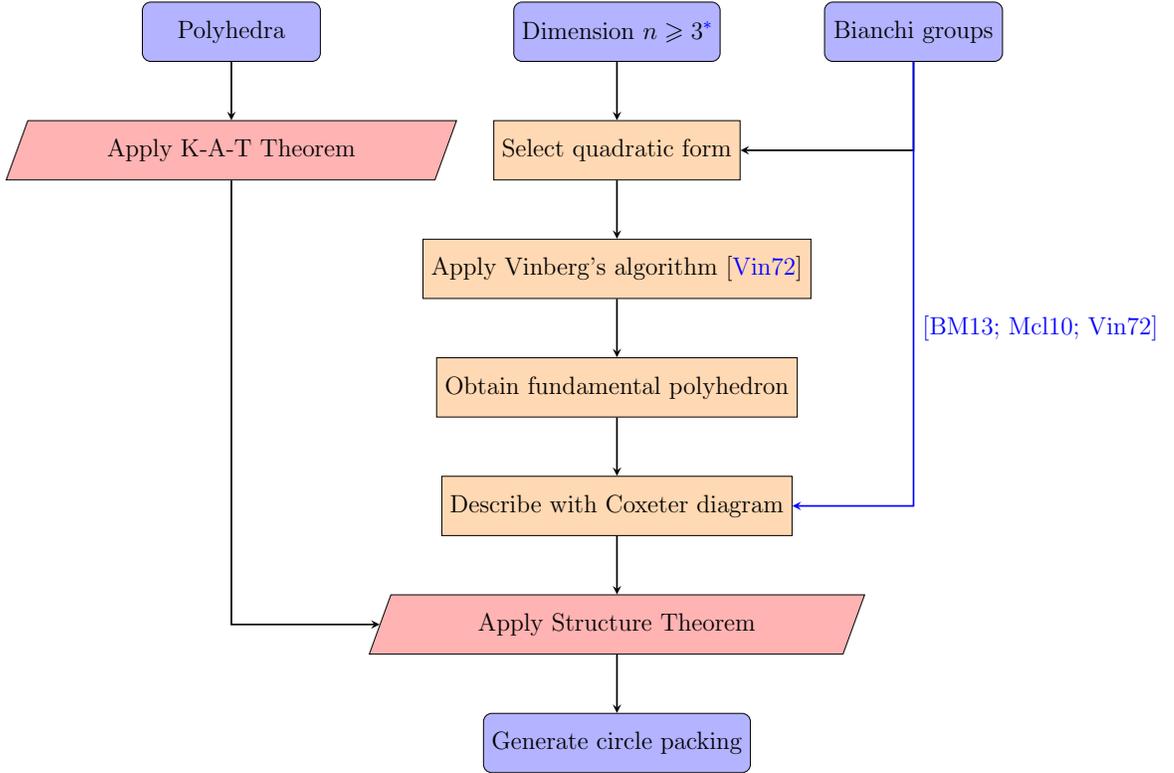
3. General Methods

3.1. **Producing crystallographic packings.** Every cluster $\mathcal{C}$ identified from one of our three sources above was used to produce a crystallographic packing by applying Theorem 4. To do so, all circles in the identified $\mathcal{C}$ were reflected about circles in $\widehat{\widehat{\mathcal{C}}}$. For each $\mathcal{C}$, we built an inversive coordinate matrix V, which we reflected about all $\widehat{v} \in \widehat{\mathcal{C}}$ by

$$R_{\widehat{v}}(V) = VR \tag{20}$$

to obtain the inversive coordinates of the next generation of circles in the packing. To obtain further generations, each new circle produced by (20) was reflected about all $\widehat{v} \in \widehat{\mathcal{C}}$ in a similar manner, the infinite repetition of which produces a crystallographic packing.

Diagrams of the packings were produced by applying Mathematica's graphics features to the list of inversive coordinates of the packing.

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*The arrow connecting *Bianchi groups* to *Coxeter diagram* via [BM13; Mcl10; Vin72] should be taken to connect *Dimension n ⩾ 3* to *Coxeter diagram* as well. This arrow indicates that our research relied on Belolipetsky & McLeod's and Vinberg's conversions of Bianchi groups and higher dimensional forms into Coxeter diagrams, performed through the steps indicated.

3.2. **Proving integrality, nonintegrality, and superintegrality.** One feature of crystallographic packings to study is the bend of each sphere in the packing.

Definition 21 (integral, superintegral). If every sphere in a crystallographic packing has integer bend, then it is an *integral* packings. If every sphere in a superpacking has integer bend, then it is a *superintegral* packing.

Finding every integral (and superintegral) crystallographic packing is of fundamental interest, and a main objective of our study. The following lemmata outline our general methods of proving integrality, non-integrality, and superintegrality of crystallographic packings.

Lemma 22. *There is always a transformation which scales the bends of all circles in a packing by some constant.*

Proof. Let $d_1 = |z| - r$ be the point on a circle s closest to the origin, and let $d_2 = |z| + r$ be the point on s furthest from the origin. Inversion through the unit circle sends $d_1 \mapsto \frac{1}{d_1}$ and $d_2 \mapsto \frac{1}{d_2}$. Subsequent inversion through a circle of radius α centered at the origin sends $\frac{1}{d_1} \mapsto \alpha^2 d_1 = \alpha^2(|z| - r)$ and $\frac{1}{d_2} \mapsto \alpha^2 d_2 = \alpha^2(|z| + r)$. Thus, the new circle has radius $\alpha^2 r$ and any circle can be rescaled by choice of α. $\square$

As a consequence, if a packing has "bounded rational" bends—i.e., no bend in the packing has denominator greater than some upper bound—then there is a conformally equivalent integral packing.

Lemma 23. *If all bend matrices of a cluster $\mathcal{C}$ are integral and the bend of each circle in $\mathcal{C}$ is rational, then the packing generated by $\widehat{\mathcal{C}}$ on $\mathcal{C}$ is integral.*

Proof. If all bends in $\mathcal{C}$ are integral, then the action $BV = VR$ is always an integral linear combination of integers, and therefore integral. Otherwise, Lemma 22 allows a rescaling of the bends to integers. $\square$

Note that Lemma 23 can hold even if the bend of each circle in $\mathcal{C}$ is irrational in the case that the bends can be uniformly rescaled by Lemma 22 to achieve integrality.

Lemma 24. *Let V be an $m \times (n + 2)$ matrix of inversive coordinates corresponding to a cluster $\mathcal{C}$ of m circles. If there exists a square matrix g satisfying $gV = 0$ with a nonrational (implying also nonintegral) linear relationship between some two entries in any row, then $\mathcal{C}$ cannot be integral.*

Proof. A nonintegral relation between the entries of the bend matrices precludes the possibility of an integral packing, since the packing is entirely generated by reflections, namely, multiplication with its bend matrices. $\square$

The following theorem from [KN17] relates superintegrality to arithmeticity as defined by Vinberg's arithmeticity criterion [Vin67].

Theorem 25. *If a packing is superintegral, then the group $\widetilde{\Gamma}$ generated by reflections through $\mathcal{C}$ and $\widehat{\mathcal{C}}$ is arithmetic.*

4. Polyhedral Packings

A version of the Koebe-Andreev-Thurston theorem allows polyhedra (equivalently, 3-connected planar graphs) to be realized as circle packings.

Theorem 26 (Koebe-Andreev-Thurston Circle Packing Theorem). *Every 3-connected planar graph can be realized as a polyhedron with a midsphere, and this realization is unique up to conformal equivalence.*

Here, a *midsphere* is an inscribed sphere tangent to every edge of a polyhedron Π. Its dual polyhedron $\widehat{\Pi}$ has the same midsphere. A realization of Π, $\widehat{\Pi}$, and their midsphere gives rise to two sets of circles (see Figure 1) which pass through edge tangency points and have normal vectors along the rays connecting vertices (of both Π and $\widehat{\Pi}$) to the center of the midsphere.

Stereographic projection of these circles onto $\mathbb{R}^2 \cup \{\infty\}$ yields a collection of circles which by Theorem 4 can be viewed as a cluster-cocluster pair $\mathcal{C}$, $\widehat{\mathcal{C}}$ giving rise to a circle packing: call circles in $\mathcal{C}$ those centered around vertices of Π and circles in $\widehat{\mathcal{C}}$ those centered around vertices of $\widehat{\Pi}$. Any two circles in $\mathcal{C}$ are either tangent or disjoint, and every circle in $\widehat{\mathcal{C}}$ is only tangent, disjoint, or orthogonal to circles in $\mathcal{C}$. The packing produced by a polyhedron Π is called $\mathscr{P}$; similarly a superpacking is called $\widetilde{\mathscr{P}}$.

Previous work has classified certain types of polyhedra, for example *uniform polyhedra*: those whose faces are regular polygons and which are vertex-transitive. From Kontorovich and Nakamura we know the following theorem.

Theorem 27. *The only integral uniform polyhedra are:*

- *(Platonic) tetrahedron, octahedron, cube;*
- *(Archimedean) cuboctahedron, truncated tetrahedron, truncated octahedron;*
- *(prisms/antiprisms) 3,4,6-prisms, 3-antiprism.*

4.1. **Methods.** We were able to systematically generate polyhedron raw data using the program `plantri`. Mathematica programs turned data into packings using techniques outlined in [BS04] (see also [Zie04; Riv86; Riv94; Col91]). Currently all polyhedra are documented on vertices $n \leq 7$, with some additional larger regular polyhedra.

We identify 2 broad categories of polyhedra which branch into 4 total smaller subcategories: integral-superintegral, integral-nonsuperintegral, nonintegral-rational, and nonintegral-nonrational. To more accurately define the relationships between polyhedra, we introduce a *gluing operation*.

Definition 28 (gluing operation). Polyhedra can be glued along faces or vertices. Let A be a polyhedron with vertex set V_A, edges E_A, and faces F_A. Similarly let $B = \{V_B, E_B, F_B\}$. A face-face gluing operation is only valid if two n-gon faces are
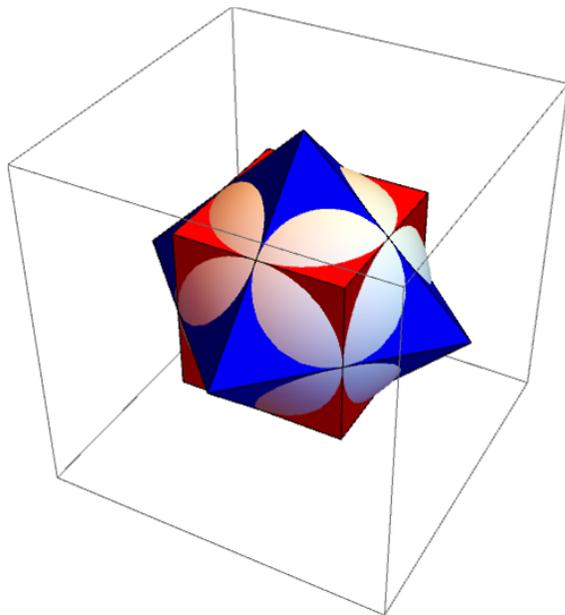
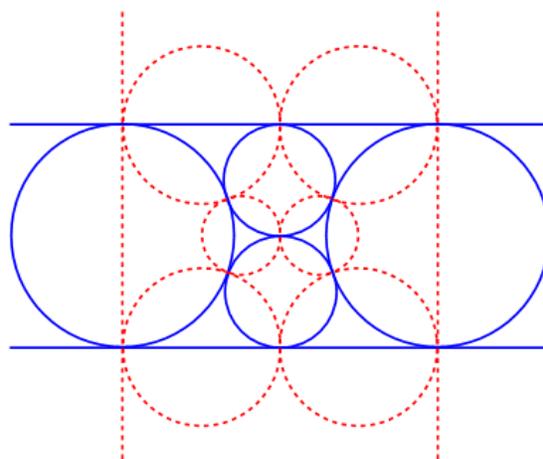FIGURE 1. Octahedron (blue), its dual (red cube), and midsphere (grey)

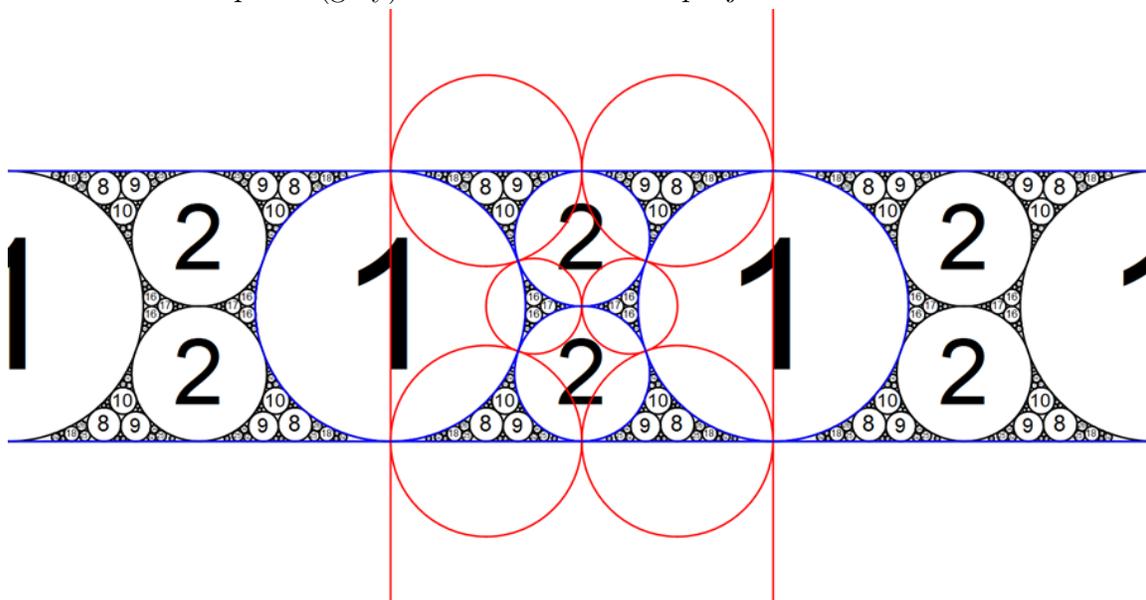FIGURE 2. Stereographic projection onto $\mathbb{R}^2$

FIGURE 3. Packing with bends

equivalent: the same types of faces, in the same order, are adjacent to both. A vertex-vertex gluing operation is only valid if two vertices of degree n are equivalent: they lie on the same type of faces, in the same order.

- To glue faces $f_a \in A$ and $f_b \in B$: let f_a, f_b be n-gons bounded by vertices $\{v_{a_1}, \ldots, v_{a_n}\}, \{v_{b_1}, \ldots, v_{b_n}\}$ and edges $\{e_{a_1}, \ldots, e_{a_n}\}, \{e_{b_1}, \ldots, e_{b_n}\}$. Vertices and edges must be glued together in a one-to-one mapping with stretching distortions only in the plane of faces f_a, f_b, which are ommitted from the final polyhedron.
- To glue two vertices of equal degree n: let $v_a \in A$ and $v_b \in B$ have edges $\{e_{a_1}, \ldots, e_{a_n}\}$ and $\{e_{b_1}, \ldots, e_{b_n}\}$. Edges are joined in a one-to-one mapping creating new faces bounded by preexisting edges from A and B such that new face m is bounded by the union of all edges on faces $f_{a_m} \in A$ and $f_{b_m} \in B$, and dropping both v_a and v_b in the final polyhedron.

(a) (b) (c) (d)
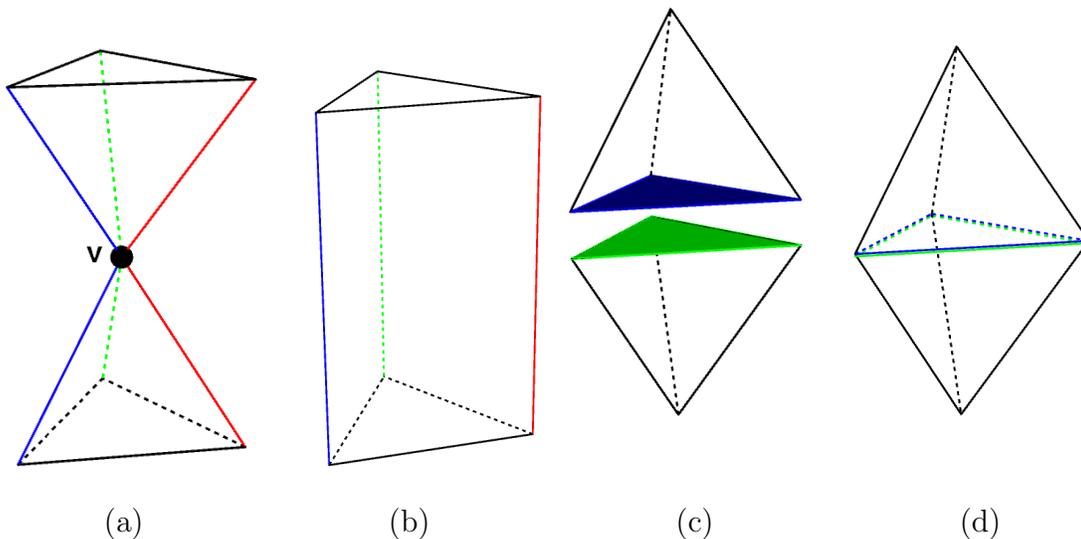
FIGURE 4. Gluing two tetrahedra at vertex v (a) to produce a triangular prism (b), gluing two tetrahedra along the blue and green faces (c) to produce triangular bipyramid (d)

4.2. **Integral Polyhedra.** A polyhedron is called *integral* if it has some associated integral packing.

Theorem 29. *There are exactly 4 unique, nondecomposable (not the result of some series of gluing operations) integral polyhedra with $n \leq 7$ vertices: tetrahedron, square pyramid, hexagonal pyramid, and unnamed 6v7f_2. We call them seed polyhedra, as in [KN17].*

FIGURE 6. (a) Tetrahedron, (b) square pyramid, (c) hexagonal pyramid, and (d) 6v7f_2

This proof of this theorem relies on the following lemma.

Lemma 30. *A gluing operation of A onto B yields a polyhedron with strictly more vertices, edges, and faces than either A or B. In particular,*

- *gluing A and B along an n-gon face yields polyhedron C such that $|V_C| = |V_A| + |V_B| - n$, $|E_C| = |E_A| + |E_B| - n$, $|F_C| = |F_A| + |F_B| - 2$, and*
- *gluing A and B at a vertex of degree n yields polyhedron C such that $|V_C| = |V_A| + |V_B| - 2$, $|E_C| = |E_A| + |E_B| - n$, $|F_C| = |F_A| + |F_B| - n$.*

Proof of Theorem 29. Aside from the polyhedra specifically mentioned in Theorem 29, only 9 of size $n \leq 7$ vertices are integral. Methods described in Lemma 24 and Lemma 33 are used to show that all others are not integral; the series of gluings used to construct the 9 others are detailed in §A. What remains is to show that the tetrahedron, square pyramid, hexagonal pyramid, and 6v7f_2 cannot be constructed from a series of gluing operations.

- A tetrahedron is the smallest possible polyhedron; by Lemma 30 it cannot be the result of gluings.
- The square pyramid is only larger than a tetrahedron, but gluing two tetrahedra yields (along a face) $|V| = 5$, $|E| = 9$, $|F| = 6$ or (along a vertex) $|V| = 6$, $|E| = 9$, $|F| = 5$; a square pyramid has 8 edges.
- 6v7f_2 does not arise from gluing two tetrahedra (above). Gluing two square pyramids yields either $|V| = 8$ or $|F| = 8$; gluing a tetrahedron to a square pyramid (by symmetry) has only two possibilities, one with $|V| = 7$, one shown in [insert figure].

- Hexagonal pyramid is not the product of any gluings described above; the addition of 6v7f_2 to possible generators cannot contribute to its construction because they share the same number of faces.

□

We can further distinguish integral polyhedra by studying the stronger condition of superintegrality.

4.2.1. *Integral-Superintegral Polyhedra.* Of the four known seed polyhedra, two are also superintegral. The tetrahedron and square pyramid (as well as the other documented superintegral polyhedra) can be proved superintegral by an extension of Lemma 23.

Aside from the polyhedra that have been documented, a theorem from [KN17] guarantees the existence of additional superintegral polyhedra.

Theorem 31. *Let A be a superintegral polyhedron. If A' is obtained by performing valid gluing operations on A, then*

$$\widetilde{\mathscr{P}}(A') \subset \widetilde{\mathscr{P}}(A)$$

Proof. Let $A = \{V_A, E_A, F_A\}$ be superintegral and $A' = \{V_{A'}, E_{A'}, F_{A'}\}$ be the polyhedron obtained by gluing $B = \{V_B, E_B, F_B\}$ to A along vertex v. By definition, $V_{A'} = (V_A \cup V_B) \backslash v$, but in particular each vertex in V_B is obtained by action of R_v on some vertex in V_A. Similarly, each face in $V_{A'}$ is either already in V_A or the result of R_v applied to a face in V_A.

By Lemma 32, all reflections in $A' \backslash B$ can be rewritten as a composition of reflections in A, since all elements of A' are simply reflections in A applied to elements of A. □

Lemma 32. *Let $\widehat{v}_1 = v_1 R_v$. Then a reflection about $\widehat{v}_1$ is equivalent to a series of reflections about v and v_1, in particular*

$$R_{\widehat{v}_1} = R_v R_{v_1} R_v.$$

Proof.

$$
\begin{aligned}
R_{\widehat{v}_1} &= R_{v_1 R_v} \\
&= I + 2Q(v_1 R_v)^T (v_1 R_v) \\
&= I + 2Q[I + 2v^T v Q^T] v_1^T v_1 [I + 2Q v^T v] \\
&= I + 2Q v_1^T v_1 + 4Q v^T v Q v_1^T v_1 + 4Q v_1^T v_1 Q v^T v + 8Q v^T v Q v_1^T v_1 Q v^T v \\
&= R_v R_{v_1} R_v.
\end{aligned}
$$

□

By Theorem 25, the groups associated with the above described superintegral polyhedra are arithmetic. However, not all integral polyhedra are superintegral. As it turns out, the contrapositive of this theorem completely describes all known integral-not superintegral polyhedra.

4.2.2. *Integral-Not Superintegral Polyhedra.* Of the known integral seed polyhedra, the hexagonal pyramid and 6v7f_2 are not superintegral. Lemma 23 does not apply to either case as rational entries are present in one or more bend matrix, so integrality is proved by conjugation of the bend matrices. Superintegrality can be disproved by extension of Lemma 33 as well as application of Theorem 25, as the polyhedra are not arithmetic by the criterion described in [Vin67].

4.3. **Nonintegral Polyhedra.** Aside from the four seed polyhedra and their gluings, we have documented many more polyhedra which are not integral. These can be understood in two broad subcategories.

4.3.1. *Rational-Nonintegral Polyhedra.* Of the nonintegral polyhedra, some (7v8f_9, 7v9f_8) have exclusively rational packings: rather than all bends being integral, they are all rational. An intuitive step would be to apply Lemma 22 and find a conformally equivalent integral packing, however these packings cannot be rescaled, a result of the following lemma.

Lemma 33. *Let $\{B_1, \ldots, B_n\}$ be bend matrices (see Definition 18) of Π and B be any product of $\{B_1, \ldots, B_n\}$. If there is an entry in B^n whose denominator grows without bound as $n \to \infty$, then Π cannot be integral.*

4.3.2. *Nonrational-Nonintegral Polyhedra.* All polyhedra which do not fit in one of the previous categories can be proved nonintegral by Lemma 24.

5. Bianchi Group Packings

Definition 34 (Bianchi group)**.** A Bianchi group $Bi(m)$ is the set of matrices

$$SL_2(\mathcal{O}_m) \rtimes \langle \tau \rangle, \tag{35}$$

where $\mathcal{O}_m$ indicates the ring of integers $\mathbb{Z}[\sqrt{-m}]$, m is a positive square-free integer, and τ is a second-order element that acts on $SL_2(\mathcal{O}_m)$ as complex conjugation [Vin90; BM13].

These groups can also be viewed as discrete groups of isometries. [Bia92] found that $Bi(m)$ is reflective—meaning that it is generated by a finite set of reflections—for $m \leqslant 19, m \notin \{14, 17\}$ [Bia92]; this list is complete ([BM13]).

[Vin72] contains an algorithm on general quadratic forms, which takes the integral automorphism group of a quadratic form and halts if the reflection group is finitely generated, producing its fundamental polyhedron. [Mcl13] applied this algorithm to the reflective *extended Bianchi groups* $\widehat{Bi}(m)$—the maximal discrete extension of $Bi(m)$ (cf. [All66; BM13], see also [Ruz90; Sha90]). To be discrete, the fundamental polyhedron produced in the case of a finitely generated reflection group must be a Coxeter polyhedron, and thus a Coxeter diagram can be drawn. [Mcl13] provides the

roots bounding the fundamental polyhedron obtained from Vinberg's algorithm for all reflective extended Bianchi groups.[*]

5.1. **Determining the clusters.** We computed the Gram matrix for each $\widehat{Bi}(m)$. We then iterated through the Gram matrix of each $\widehat{Bi}(m)$ to identify all existing clusters and subgroups thereof.[†] By Theorem 4, a Gram matrix can be used to identify clusters by identifying those rows whose entries G_i exclusively satisfy the condition $|G_i| \geqslant 1$ or $G_i = 0$.

Every cluster $\mathcal{C}$ within each $\widehat{Bi}(m)$ was then used to produce a crystallographic packing by applying Theorem 4 via the methods outlined in §3.1 above.

---

[*]Corrections for errors in [Mcl13]'s listings of roots can be found in §C.

[†]We excluded all clusters wherein two vertices were orthogonal to each other due to redundancy in crystallographic circle packings.

$$\begin{pmatrix} 0 & 0 & -1 & 0 \\ 8 & 6 & 3 & 2\sqrt{10} \end{pmatrix}$$

(a)

$$\begin{pmatrix} 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \\ \sqrt{10} & 0 & 0 & 1 \\ -1 & 1 & 0 & 0 \\ 2\sqrt{2} & \sqrt{2} & 0 & \sqrt{5} \\ 3\sqrt{10} & 3\sqrt{10} & \sqrt{10} & 9 \\ 4\sqrt{10} & 4\sqrt{10} & 2\sqrt{10} & 11 \end{pmatrix}$$
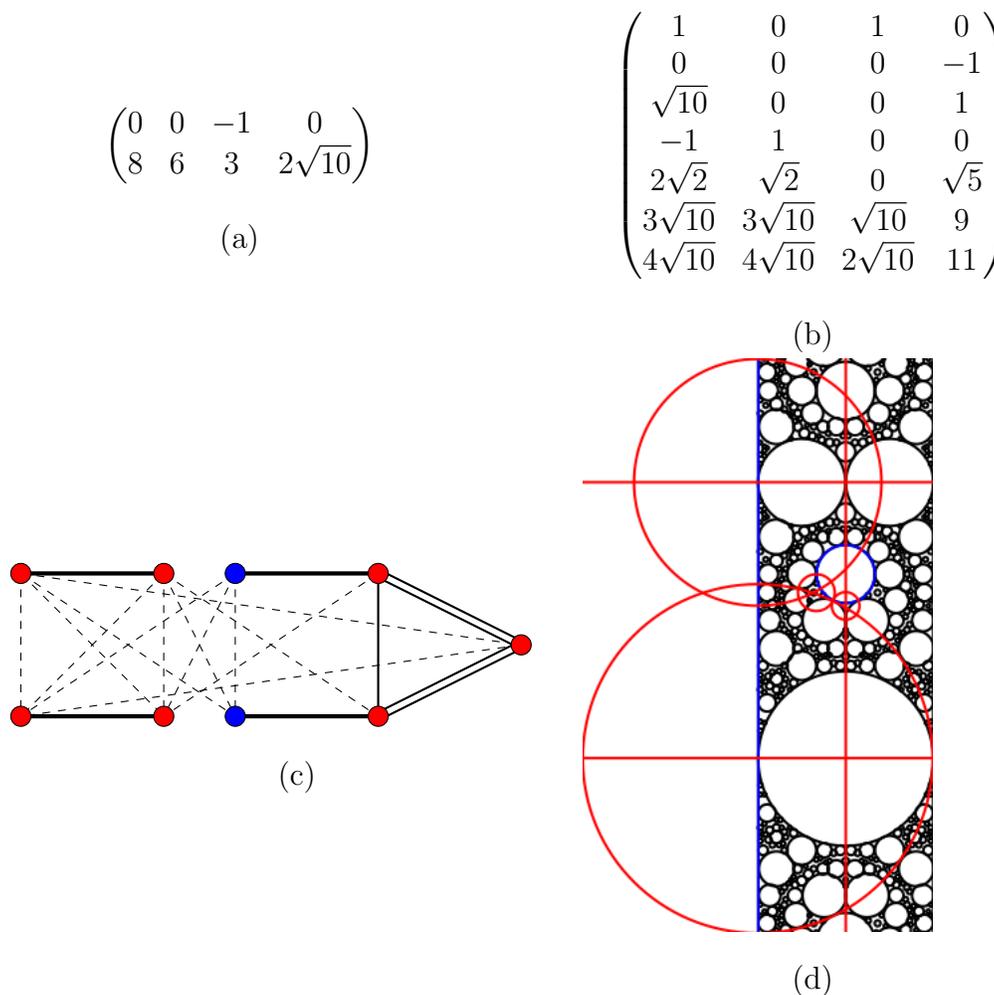
(b)

(c)

(d)

FIGURE 7. (a) Coordinates of cluster $\{1,7\}$ for $\widehat{Bi}(10)$. (b) Corresponding cocluster coordinates. (c) Coxeter diagram for $\widehat{Bi}(10)$, with cluster $\{1,7\}$ highlighted in blue. (d) Diagram of $\widehat{Bi}(10)$ cluster $\{1,7\}$ packing, cluster in blue, with cocluster in red.

5.2. **Results.** All crystallographic packings that arise from the extended Bianchi groups have been documented, namely all packings from $\widehat{Bi}(m)$ for $m = 1, 2, 5, 6,$ 7, 10, 11, 13, 14, 15, 17, 19, 21, 30, 33, 39, each with their corresponding Coxeter diagram, inversive coordinates matrix, Gram matrix, and diagrams of all possible cluster packings.*

---

*Note that $\widehat{Bi}(3)$ as recorded in [BM13] does not yield a crystallographic packing. See §6.2.2 for treatment of $\widehat{Bi}(3)$.

All extended Bianchi group packings are determined to be arithmetic. We then classified all integral and non-integral extended Bianchi group packings.[*] There are 145 integral $\widehat{Bi}(m)$ packings and 224 non-integral $\widehat{Bi}(m)$ packings, a complete list of which can be found in §D. Our first step was to compute the orbit of each packing up to some generation, and empirically conjecture whether or not the packing would be integral based on the bends produced. We then rigorously proved such conjectures, as described in the following sections.

$$\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 2 & 2 & 2 & -1 \end{pmatrix} \cdot \begin{pmatrix} 0 & 0 & 0 & -1 \\ 2 & 0 & 0 & 1 \\ 0 & 2 & 0 & 1 \\ 2 & 2 & 2 & 1 \end{pmatrix}$$
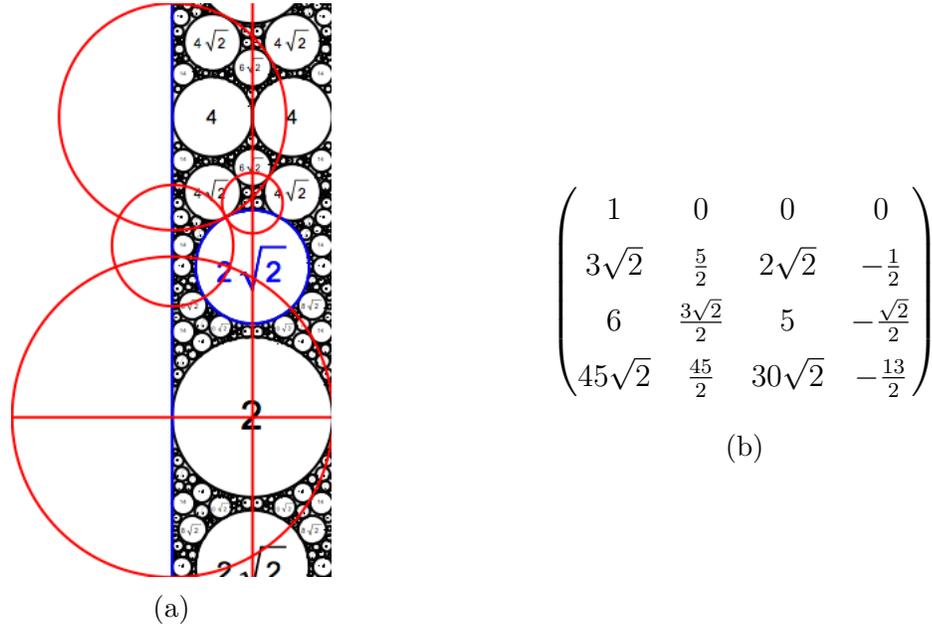
(b)

(a)

FIGURE 8. (a) Integral packing generated from $\widehat{Bi}(1)$ cluster $\{3\}$, depicted in blue, along with cocluster, depicted in red. Numbers indicate bends. (b) BV for $\widehat{Bi}(1)$ cluster $\{3\}$ and its orbit, demonstrating integrality of the packing.

5.2.1. *Integral Bianchi group packings.* To prove integrality for Bianchi group packings, Lemma 22 is first applied when necessary to rescale inversive coordinates of the clusters to integrality.

In the cases where the associated bend matrices B_i for a given Bianchi packing are integral, integrality is immediately proven via Lemma 23. (See Figure 9 for an example.) In the cases where the associated B_i's are not integral, integrality can still be proven through Lemma 23 by first calling upon the following lemma, the proof of which relies on the integrality of all right-acting reflection matrices R associated with each $\widehat{Bi}(m)$ packing.[†]

---

[*]Note that this is up to rescaling via circle inversion.

[†]As proven in [BM13], there exist finitely-many $\widehat{Bi}(m)$ packings. We have generated reflection matrices R for all such packings, and determined that every associated R matrix is integral. Note

Lemma 36. *The product of any bend matrices of an integral Bianchi group packing is "bounded rational."*

Proof. By (19), for V a $(n+2) \times (n+2)$ full-rank matrix of inversive coordinates, the right-acting reflection matrix R can be expressed as

$$R = V^{-1}BV. \tag{37}$$

Since B is simply a left-acting reflection matrix, action of B_i on $B_j V$ is simply further reflection of V, and hence $B_i B_j$ is a proper bend matrix. Similarly, R can be broken into a series of reflection matrices $R_1 \cdots R_i$. Thus, we have

$$V R_1 \cdots R_i V^{-1} = B_i \cdots B_1. \tag{38}$$

Suppose that $B' = B_i \cdots B_1$ has unbounded denominators. Since V, V^{-1} are fixed matrices, this would imply that $R' = R_1 \cdots R_i$ has unbounded denominators as well. However, as each R_j is known to be integral, R' is definitely integral. Therefore, B' cannot have unbounded denominators. $\square$

Lemma 36 implies that it is possible to clear the denominators of any B' through some rescaling of V. Because there are no irrational entries in the integral Bianchi group bend matrices, and all bends of the cluster can be rescaled to integrality by Lemma 22, this proves that all bends produced by $B'V$ will then be integral.

---

that an alternative proof of integrality for Bianchi groups whose bend matrices are not integral is implied by [KN17]'s discussion of arithmeticity of the supergroup of a superintegral packing.

(a)

$$\begin{pmatrix} 1 & 0 & 0 & 0 \\ 3\sqrt{2} & \frac{5}{2} & 2\sqrt{2} & -\frac{1}{2} \\ 6 & \frac{3\sqrt{2}}{2} & 5 & -\frac{\sqrt{2}}{2} \\ 45\sqrt{2} & \frac{45}{2} & 30\sqrt{2} & -\frac{13}{2} \end{pmatrix}$$

(b)

FIGURE 9. (a) Nonintegral packing generated from $\widehat{Bi}(14)$ cluster {1,8}, depicted in blue, along with cocluster, depicted in red. Numbers indicate bends. Note mixture of integral and nonintegral bends, indicating that this packing cannot be rescaled to integrality. (b) One of the associated bend matrices B. Note the existence of a nonlinear relation in each row except the first.

5.3. **Nonintegral Bianchi group packings.** While all Bianchi groups give rise to integral packings, there exist packings whose cluster circles cannot be rescaled to integrality which will produce nonintegral packings. For instance, the packing produced by $\widehat{Bi}(17)$ with the cluster of vertices {8, 13} has both a bend of $\sqrt{34}$ (vertex 8) and a bend of $\sqrt{17}$ (vertex 13); clearly this cluster cannot be rescaled in such a way that clears both bends to integrality.

To prove nonintegrality of such packings, we built an over-determined matrix V from the inversive coordinates of the cluster, with supplementary inversive coordinates from the orbit added if necessary. We then applied Lemma 24. In every non-integral packing, the solution set for g contained irrational coefficients, indicating an inherent nonintegral relation between the entries of the bend matrices for the packing. (See §E for an example.)

6. HIGHER DIMENSIONAL PACKINGS

We now seek to apply Theorem 4 to the finitely-many commensurability classes guaranteed by Theorem 6, as these classes are known to exist but are not guaranteed

to admit packings. There are a number of techniques available to produce such candidates. We note that all packings produced through these techniques are arithmetic, and in doing so we show that the quadratic forms in question are commensurate with a supergroup of a packing.

We consider for fixed d and n the configuration of inversive coordinates $\widetilde{V} = \{v_i\}_{i=1}^m$ and Gram matrix $G = \{g_{ij}\}_{i,j=1}^m$. (More generally, we'll consider i, j instead ranging on some explicit index set.) In Theorem 44, we show an important result that validates a technique (doubling, see Definition 43) that contributes to producing the desired packings.

Definition 39. For $1 \leqslant a, b \leqslant m$, we write $b.a$, "the action of b on a" or "b acting on a," to denote $v_a R_{v_b}$.

Note that this action is right-associative, i.e. $a.b.c = a.(b.c)$.

Lemma 40. $a.a.b = b$, as inversion is an involution.

Lemma 41. $(a.b).c = a.b.a.c$.

Proof. This is Lemma 32 under different notation. □

Definition 42 (reversing orientation)**.** Let $\widetilde{a}$ denote the oriented hypersphere a with reversed orientation, i.e. $a.a$.

6.1. **Doubling.**

Definition 43 (doubling)**.** Let $\widetilde{V}_j = \widetilde{V} \backslash \{v_j\}$ for $1 \leqslant j \leqslant m$. We say that we *double* $\widetilde{V}$ about j when we compute $\widetilde{V}^j = \widetilde{V}_j \cup j.\widetilde{V}_j$.

Doubling can be thought of as a "hyperbolic gluing" in the same vein as Definition 28 for Euclidean polyhedra, wherein a configuration is extended into $\mathbb{H}^{n+1}$ through Poincaré extension and then doubled about a face.

Theorem 44. $\left\langle \widetilde{V}^j \right\rangle < \left\langle \widetilde{V} \right\rangle$ and $\left[\left\langle \widetilde{V} \right\rangle : \left\langle \widetilde{V}^j \right\rangle \right] \leqslant \infty$.

Proof. Without loss of generality, let $j = 1$.

$\left\langle \widetilde{V}^1 \right\rangle < \left\langle \widetilde{V} \right\rangle$ follows immediately as $\left\langle \widetilde{V}^1 \right\rangle = \langle 2, \ldots, m, 1.2, \ldots, 1.m \rangle$ consists exclusively of elements of $\left\langle \widetilde{V} \right\rangle$.

We aim to show that if $\widetilde{V}$ extends through Poincaré extension to a $\mathbb{R}^n$-bounded $\mathbb{H}^{n+1}$ polytope P that touches $\mathbb{R}^n$ at finitely many cusps, then doubling about 1 also yields such a polytope P^1. This will prove the result because the quotients of the volumes equals the index – $[P^1] / [P] = \left[\left\langle \widetilde{V} \right\rangle : \left\langle \widetilde{V}^1 \right\rangle \right]$. This is because each coset represents a "copy" of P that can be mapped into P^1 through members of $\left\langle \widetilde{V} \right\rangle$, i.e. P can be thought as the gluing-together of several copies of P^1, and specifically $\left[\left\langle \widetilde{V} \right\rangle : \left\langle \widetilde{V}^1 \right\rangle \right]$-many.

Let j^+ denote the interior of v_j and let j^- denote the interior of $\widetilde{j}$. Further, let $\sqcap_1 = \bigcap_{j=2}^{m} j^+$. The interior of $\widetilde{V}^1$ is

$$
\begin{aligned}
\sqcap_1 \cap 1.\sqcap_1 =& (\sqcap_1 \cap (1^+ \cap 1.\sqcap_1)) \\
& \cup (\sqcap_1 \cap (1 \cap 1.\sqcap_1)) \\
& \cup (\sqcap_1 \cap (1^- \cap 1.\sqcap_1)), \\
\sqcap_1 \cap (1^+ \cap 1.\sqcap_1) =& ((1^+ \cap \sqcap_1) \cap (1^+ \cap 1.\sqcap_1)) \\
& \cup ((1 \cap \sqcap_1) \cap (1^+ \cap 1.\sqcap_1)) \\
& \cup ((1^- \cap \sqcap_1) \cap (1^+ \cap 1.\sqcap_1)).
\end{aligned}
$$

(45)

(46)

From the hypothesis, we have that $(1^+ \cup 1) \cap \sqcap_1$ is finite, and we further know that $1^- \cap 1^+ = \emptyset$ and is thus finite; thus, $\sqcap_1 \cap 1.\sqcap_1$ is the finite union of finite sets and is finite.

Similarly, we can prove boundedness: we know that $(1^+ \cup 1) \cap \sqcup_1$ is bounded, so substituting (46) into (45) as was implicitly done just above gets

$$
\begin{aligned}
\sqcap_1 \cap 1.\sqcap_1 =& ((1^+ \cap \sqcap_1) \cap (1^+ \cap 1.\sqcap_1)) \\
& \cup ((1 \cap \sqcap_1) \cap (1^+ \cap 1.\sqcap_1)) \\
& \cup ((1^- \cap \sqcap_1) \cap (1^+ \cap 1.\sqcap_1)) \\
& \cup (\sqcap_1 \cap (1 \cap 1.\sqcap_1)) \\
& \cup (\sqcap_1 \cap (1^- \cap 1.\sqcap_1))
\end{aligned}
$$

(47)

which is the finite union of bounded and empty sets, and hence is bounded.   □

Doubling, in certain cases, creates configurations that have clusters. However, this requires that the resultant configuration exclusively has angles of the form $\frac{\pi}{n}$, in order for there to exist a Coxeter diagram. Practically speaking:

Lemma 48. *If doubling about 1 generates a reflective group, then in $\widetilde{V}$'s Coxeter diagram, 1 is joined to other nodes exclusively by an even number of edges. (A thick line is considered to be an even number of edges.)*

Lemma 48 is used to remove mirrors in a configuration from consideration for doubling. Of course, it is also necessary that the resultant configuration generates a group of mirrors commensurate to the original group. This is the primary function of Theorem 44.

The principle of doubling was successfully applied to obtain clusters, and hence commensurability classes yielding packings, described in full in §G.1 and §G.2.

6.2. **Beyond Doubling.** Unfortunately, doubling is not a panacea, and there remain instances that are left unresolved by doubling. Here we describe instances where the Coxeter diagram obtained by Vinberg's algorithm, either in [Vin72] or [Mcl10], lack a cluster, as do all of their doublings, but there exist other subgroups that admit packings through their diagrams. We begin our siege on these other diagrams with a lemma useful in proofs to come.

Lemma 49. *The n-dimensional oriented hypersphere specified by $\left(\widehat{b}, b, bz\right)$ has interior given by $x = (x_i)_{i=1}^n \in \widehat{\mathbb{R}^{n*}}$ satisfying*

$$0 < \begin{cases} bz \cdot x - \frac{1}{2}\widehat{b} & b = 0 \\ \left(\frac{1}{b^2} - |z - x|^2\right) \operatorname{sign} b & b \neq 0 \end{cases}$$

where $v \cdot w = v I_n w^T$, i.e. the "standard" dot product.

Proof. **Case $b = 0$.** We know that the hyperplane has normal vector given by $\widehat{n} = bz$, from Lemma 13. We first consider the case $\widehat{b} = 0$, i.e. the codimension-1 hyperplane passes through the origin (by Lemma 13, as in that context we would have $|P| = 0 \implies P = 0$). The half-space "on the same side as" $\widehat{n}$ is characterized by $\widehat{n} \cdot x > 0$. Consider now the case $\widehat{b} \neq 0$. Consider a translation of space sending $P \mapsto 0$ and $x \mapsto x'$. Then it is clear that we must have $\widehat{n} \cdot x' > 0$ in this mapping of space (as the plane now passes through the origin). Of course, this mapping is simply defined as $x \mapsto x' = x - P$. P can be computed as $\frac{1}{2}\widehat{b}\widehat{n}$, since P is the nearest point to the origin and hence the segment connecting 0 and P is perpendicular to the plane, thus parallel to $\widehat{n}$. Therefore, we have

$$0 < \widehat{n} \cdot \left(x - \frac{1}{2}\widehat{b}\widehat{n}\right) = \widehat{n} \cdot x - \frac{1}{2}\widehat{b}\widehat{n} \cdot \widehat{n} = bz \cdot x - \frac{1}{2}\widehat{b}.$$

Case $b \neq 0$. If $b > 0$ then we are looking for the standard notion of the interior of a hypersphere, i.e. the points within r of the center. Namely,

$$r = \frac{1}{b} > |z - x| > 0 \implies \frac{1}{b^2} > |z - x|^2 \implies \frac{1}{b^2} - |z - x|^2 > 0.$$

If $b < 0$ then we are looking for the complement of $\overline{B_z\left(1/|b|\right)}$ (the closed ball centered at z with radius $1/|b|$), i.e. x must satisfy

$$0 < r = -\frac{1}{b} < |z - x| \implies \frac{1}{b^2} < |z - x|^2 \implies \frac{1}{b^2} - |z - x|^2 < 0.$$

We now see that both cases of b's sign are captured by $\left(\frac{1}{b^2} - |z - x|^2\right) \operatorname{sign} b > 0$. $\quad\square$

Definition 50. In the context of $\widetilde{V} = \{v_i\}_{i=1}^m$ a configuration of inversive coordinates generating $\left\langle \widetilde{V} \right\rangle \leqslant \operatorname{Isom}(\mathbb{H}^{n+1})$ of finite index where $\mathbb{H}^{n+1}$ is viewed as arising from

---

*In the case of $x = \infty$ we can safely view x as the n-tuple with each entry equal to ∞.

$-dx_0^2 + \sum\limits_{i=1}^{n} x_i^2$, and for $1 \leqslant j \leqslant m$, we write $(d.n.j)$ to denote the application of Lemma 49 to v_j.

6.2.1. $d = 3, n = 3$. We obtain the following Coxeter diagram
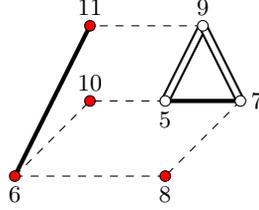
arising from the configuration

	$\widehat{b}$	b	bx	by	def'd as:
5	-2	0	$-\frac{\sqrt{3}}{2}$	$-\frac{1}{2}$	$\widetilde{3}$
6	0	0	$-\frac{1}{2}$	$\frac{\sqrt{3}}{2}$	1.2
7	6	0	$\frac{\sqrt{3}}{2}$	$\frac{1}{2}$	$3.1.2.1.2.1.3$
8	$\sqrt{2}$	$\frac{\sqrt{2}}{2}$	$\frac{\sqrt{6}}{2}$	$-\frac{\sqrt{2}}{2}$	$1.3.4$
9	$\sqrt{2}$	$\frac{\sqrt{2}}{2}$	$\frac{\sqrt{6}}{2}$	$\frac{\sqrt{2}}{2}$	3.4
10	$5\sqrt{2}$	$\frac{\sqrt{2}}{2}$	$\sqrt{6}$	0	$3.1.2.1.3.4$
11	$2\sqrt{3}$	0	$\frac{1}{2}$	$-\frac{\sqrt{3}}{2}$	$1.2.3.1.2$

$$(51)$$

Lemma 52. (51) *has empty interior in* $\mathbb{R}^2$, *and extends by Poincaré extension to a hyperbolic polytope of finite volume.*

Proof. We first show that the configuration has empty interior, by supposing towards contradiction that $x_1 \geqslant 0$:

$$1 - \frac{\sqrt{3}}{2}x_1 - \frac{1}{2}x_2 > 0 \tag{3.3.5}$$

$$x_2 > 2 + \sqrt{3}x_1 \geqslant 2$$

$$2 - (2\sqrt{3} - x_1)^2 - x_2^2 > 0 \tag{3.3.10}$$

$$2 > (2\sqrt{3} - x_1)^2 + x_2^2 \geqslant x_2^2$$

$$\geqslant 2^2,$$

a contradiction. Hence in the supposed mutual interiors of the configuration, $x_1 < 0$. However, again following $(3.3.10)$, we find

$$2 > (2\sqrt{3} - x_1)^2 + x_2^2 \geqslant (2\sqrt{3} - x_1)^2$$
$$> (2\sqrt{3})^2,$$

a further contradiction. Hence no point $(x_1, x_2) \in \mathbb{R}^2$ lies in the mutual interior of the specified configuration.

(3.3.10) also gives bounds for each coordinate: $2\sqrt{3} - \sqrt{2} < x_1 < 2\sqrt{3} + \sqrt{2}$ and $|x_2| < \sqrt{2}$. Since the intersection of the respective Poincaré extensions of the circles is bounded and does not meet the boundary of $\mathbb{H}^3$, it must be of finite volume. $\square$

This admits a packing through e.g. $\{6\}$, the cluster consisting just of the vector $\begin{pmatrix} 0 & 0 & -\frac{1}{2} & \frac{\sqrt{3}}{2} \end{pmatrix}$. This was uncovered by analyzing by hand the orbit of $\{1, 2, 3, 4\}$ acting on itself.
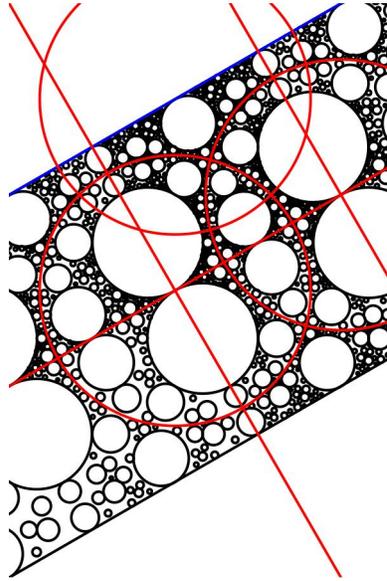
FIGURE 10. A packing arising from cluster $\{6\}$ in §6.2.1.

6.2.2. $\widehat{Bi}(3)$. [KN17] includes the following useful fact:

Lemma 53. *The Coxeter diagram for* $\widehat{Bi}(3)$,
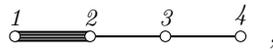
admits the subgroup corresponding to
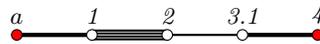
for $a = (((3.2).1).4).((3.2).1) = 3.2.3.1.3.2.3.4.3.2.3.\widetilde{1} = (3.2.3.1).4.(3.2).\widetilde{1}.$
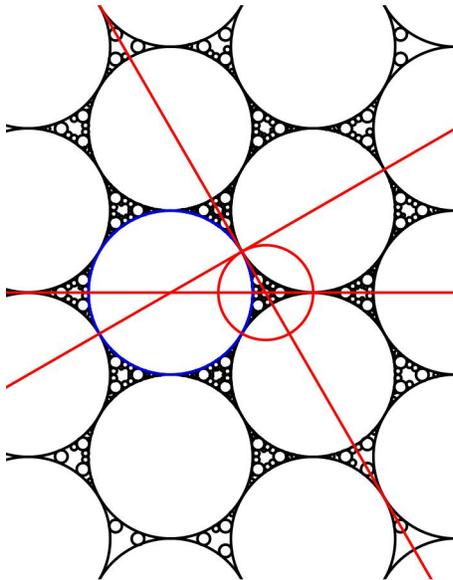
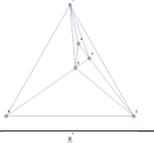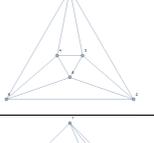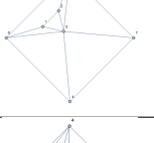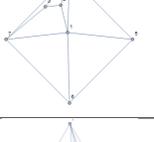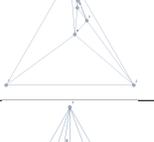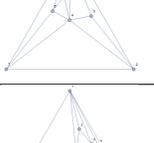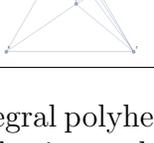FIGURE 11. A packing arising from cluster $\{4\}$ in §6.2.2.

In each of the transformations in §G.3–§G.8, Lemma 53 was applied to a subdiagram having the form $\overset{1}{\circ}\!\!\!\equiv\!\!\!\overset{2}{\circ}\!\!-\!\!\overset{3}{\circ}\!\!-\!\!\overset{4}{\circ}$ to obtain $\overset{a}{\circ}\!\!-\!\!\overset{1}{\circ}\!\!\!\equiv\!\!\!\overset{2}{\circ}\!\!-\!\!\overset{3.1}{\circ}\!\!-\!\!\overset{4}{\circ}$. Another transformation was then applied to the remaining $m - 4$ mirrors in the configuration such that the resultant configuration had a cluster and corresponded to a finite-volume $\mathbb{H}^{n+1}$ polytope, provably so by Lemma 49.

For each configuration provided in §G, the numbering implicitly referenced comes from the configurations obtained through Vinberg's algorithm in [Vin72] and [Mcl10], and listed at `math.rutgers.edu/~alexk/crystallographic`.

6.3. **Unresolved questions.** This work has shown that for $1 \leqslant d \leqslant 3$, the only cases not known to admit a commensurability class of packings are in $d = 1$, $n > 3$. Therefore the immediate next steps would be to consider those cases, as well as $d > 3$. The data for $d > 3$ can be found in [Mcl13].

APPENDIX A. INTEGRAL POLYHEDRA

A.1. **Construction of Integral Polyhedra.** All known integral polyhedra which are not one of the four seed polyhedra can be constructed by gluings of seed polyhedra. For ease of notation, we let t = tetrahedron and s = square pyramid. $\mathscr{V}^+$ is a vertex gluing and $\mathscr{F}_n^+$ is a face gluing along an n-gon face.

Name	Planar Graph	Construction
Triangular bipyramid		$t \, \mathscr{F}^+ \, t$
6v8f_1		$t \, \mathscr{F}^+ \, t \, \mathscr{F}^+ \, t$
Octahedron		$s \, \mathscr{F}_4^+ \, s$
Elongated triangular pyramid		$t \, \mathscr{V}^+ \, t \, \mathscr{F}^+ \, t$
7v8f_6		$s \, \mathscr{F}_3^+ \, s$
7v8f_7		$s \, \mathscr{F}_3^+ \, s$
7v10f_1		$t \, \mathscr{F}^+ \, t \, \mathscr{F}^+ \, t$
7v10f_2		$t \, \mathscr{F}^+ \, t \, \mathscr{F}^t \, t$
7v10f_3		$t \, \mathscr{F}^+ \, t \, \mathscr{F}^+ \, t$
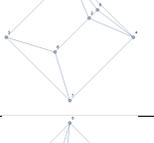
A.2. **Proving integrality.** Most of the integral polyhedra which have been identified can be proven integral by Lemma 23. The integral bend matrices associated to every such polyhedron can be found on our website. The only two which cannot be proven integral in this way are the hexagonal pyramid and 6v7f_2. The bend matrices associated to these polyhedra are rational but not strictly integral, so we must verify that the fractional components of the bends can always be cleared by rescaling. This

process is currently done in an ad hoc way by inspection of how the rational entries
of a bend matrix change under multiplication with other bend matrices.

Appendix B. Nonintegral Polyhedra

All nonintegral polyhedra can be proved as such in one of two ways: all nonintegral-nonrational by Lemma 24 and all nonintegral-rational by Lemma 33. We give an example of each below.

B.0.1. *Nonintegral-nonrational.* The following is the general form of any matrix in the cokernel of V for the polyhedron 6v7f_1.

$$\begin{pmatrix}
\alpha b_{12} - \beta b_{13} & b_{12} & b_{13} & \gamma b_{12}\delta b_{13} & \sqrt{2}\,(b_{12} + b_{13}) & \alpha\,(b_{12} + b_{13}) \\
\alpha b_{22} - \beta b_{23} & b_{22} & b_{23} & \gamma b_{22}\delta b_{23} & \sqrt{2}\,(b_{22} + b_{23}) & \alpha\,(b_{22} + b_{23}) \\
\alpha b_{32} - \beta b_{33} & b_{32} & b_{33} & \gamma b_{32}\delta b_{33} & \sqrt{2}\,(b_{32} + b_{33}) & \alpha\,(b_{32} + b_{33}) \\
\alpha b_{42} - \beta b_{43} & b_{42} & b_{43} & \gamma b_{42}\delta b_{43} & \sqrt{2}\,(b_{42} + b_{43}) & \alpha\,(b_{42} + b_{43}) \\
\alpha b_{52} - \beta b_{53} & b_{52} & b_{53} & \gamma b_{52}\delta b_{53} & \sqrt{2}\,(b_{52} + b_{53}) & \alpha\,(b_{52} + b_{53}) \\
\alpha b_{62} - \beta b_{63} & b_{62} & b_{63} & \gamma b_{62}\delta b_{63} & \sqrt{2}\,(b_{62} + b_{63}) & \alpha\,(b_{62} + b_{63})
\end{pmatrix}$$

$\alpha, \beta, \gamma, \delta$ are all irrational constants, allowing Lemma 24 to be applied.

B.0.2. *Nonintegral-Rational.*

$$\text{7v9f\_8: } (4.9)^n$$

$$\begin{pmatrix}
0 & 0 & \frac{25^{1-n}}{16} - \frac{25^n}{16} & 0 & 2\,5^{-2n} + \frac{25^n}{8} - \frac{31}{40} & -\frac{4}{5} & -\frac{7\,5^{-2n}}{16} + \frac{5^{2n}}{16} + \frac{9}{40} \\
0 & 0 & \frac{25^{1-n}}{16} - \frac{25^n}{16} & 0 & 2\,5^{-2n} + \frac{25^n}{8} - \frac{49}{40} & \frac{4}{5} & -\frac{7\,5^{-2n}}{16} + \frac{5^{2n}}{16} - \frac{9}{40} \\
0 & 0 & \frac{25^{1-n}}{18} - \frac{7\,25^n}{18} & 0 & \frac{16\,25^{-n}}{9} + \frac{7\,25^n}{9} - \frac{23}{9} & 0 & \frac{7\,25^n}{18} - \frac{7\,25^{-n}}{18} \\
0 & 0 & \frac{25^{1-n}}{64} - \frac{25^{n+1}}{64} & 0 & \frac{25^{-n}}{2} + \frac{25^{n+1}}{32} - \frac{23}{32} & -1 & -\frac{7\,25^{-n}}{64} + \frac{25^{n+1}}{64} + \frac{9}{32} \\
0 & 0 & 0 & 0 & 1 & 0 & 0 \\
0 & 0 & \frac{25^{1-n}}{64} - \frac{25^{n+1}}{64} & 0 & \frac{5^{-2n}}{2} + \frac{25^{n+1}}{32} - \frac{41}{32} & 1 & -\frac{7\,25^{-n}}{64} + \frac{25^{n+1}}{64} - \frac{9}{32} \\
0 & 0 & \frac{25^{1-n}}{18} - \frac{25^{n+1}}{18} & 0 & \frac{16\,5^{-2n}}{9} + \frac{25^{n+1}}{9} - \frac{41}{9} & 0 & \frac{25^{n+1}}{18} - \frac{7\,25^{-n}}{18}
\end{pmatrix}$$

The above matrix is the general form of $(4.9)^n$, where 4.9 is the product of bend matrices associated with the 4th and 9th faces in the polyhedron 7v9f_8. It has a number of entries which have denominators in the form c^n and are thus unbounded.

Appendix C. Corrections to [Mcl13]

The following corrections are stated in reference to Appendix F in [Mcl13]:
- In table F.2, vector e_4 should be $(2, 0, 0, -1)$, not $(1, 0, 0, -1)$.
- In table F.3, vector e_4 should be $(-1, 1, 0, 0)$, not $(1, 1, 0, 0)$.
- In table F.9, the self-product (e, e) of e_8 should be 2, not 26.
- In table F.16, e_3 should be $(0, 0, 0, 1)$ not $(0, 0, 1, -2)$; similarly e_4 should be $(33, 0, 0, 1)$, not $(33, 0, 1, -2)$, since $m \equiv 1 \pmod 4$.

- In table F.17, vector e_4 should be $(39, 0, -1, 2)$, not $(33, 0, -1, 2)$. The self-product (e, e) of e_3 should be 78, not 66; similarly the self-product (e, e) of e_4 should be 78, not 66.

Appendix D. Integral and non-integral Bianchi packings

The following is a complete list of all integral (145) and non-integral (224) crystallographic packings that arise from the extended Bianchi groups, referred to here as $\mathrm{Bi}(m)$:

Integral:
- $\mathrm{Bi}(1)$: {1}, {3}
- $\mathrm{Bi}(2)$: {1}, {3}
- $\mathrm{Bi}(5)$: {3}, {4}, {3,4}
- $\mathrm{Bi}(6)$: {1}, {3}, {4}, {3,4}
- $\mathrm{Bi}(7)$: {3}, {4}, {3,4}
- $\mathrm{Bi}(10)$: {1}, {3}, {4}, {7}, {8}, {9}, {1,7}, {3,4}, {3,8}, {3,9}, {4,8}, {4,9}, {8,9}, {3,4,8}, {3,4,9}, {3,8,9}, {4,8,9}, {3,4,8,9}
- $\mathrm{Bi}(11)$: {3}, {4}, {3,4}
- $\mathrm{Bi}(13)$: {3}, {4}, {9}, {10}, {3,4}, {3,9}, {3,10}, {4,9}, {4,10}, {9,10}, {3,4,9}, {3,4,10}, {3,9,10}, {4,9,10}, {3,4,9,10}
- $\mathrm{Bi}(14)$: {1}, {3}, {4}, {7}, {8}, {9}, {3,4}, {7,9}
- $\mathrm{Bi}(15)$: {3}, {4}, {7}, {8}, {3,4}, {3,7}, {3,8}, {4,7}, {4,8}, {7,8}, {3,4,7}, {3,4,8}, {3,7,8}, {4,7,8}, {3,4,7,8}
- $\mathrm{Bi}(17)$: {3}, {4}, {8}, {11}, {12}, {13}, {3,4}, {3,12}, {3,13}, {4,12}, {4,13}, {8,11}, {12,13}, {3,4,12}, {3,4,13}, {3,12,13}, {4,12,13}, {3,4,12,13}
- $\mathrm{Bi}(19)$: {3}, {4}, {3,4}
- $\mathrm{Bi}(21)$: {3}, {4}, {9}, {11}, {3,4}, {3,9}, {3,11}, {4,9}, {4,11}, {9,11}, {3,4,9}, {3,4,11}, {3,9,11}, {4,9,11}, {3,4,9,11}
- $\mathrm{Bi}(30)$: {1}, {3}, {4}, {8}, {9}, {10}, {11}, {3,4}, {8,11}
- $\mathrm{Bi}(33)$: {3}, {4}, {7}, {10}, {11}, {12}, {13}, {15}, {3,4}, {3,11}, {3,15}, {4,11}, {4,15}, {7,13}, {10,12}, {11,15}, {3,4,11}, {3,4,15}, {3,11,15}, {4,11,15}, {3,4,11,15}
- $\mathrm{Bi}(39)$: {3}, {4}, {9}, {10}, {3,4}, {9,10}

Non-integral:
- $\mathrm{Bi}(10)$: {1,8}, {1,9}, {3,7}, {4,7}, {1,8,9}, {3,4,7}
- $\mathrm{Bi}(14)$: {1,8}, {1,9}, {3,7}, {3,8}, {3,9}, {4,7}, {4,8}, {4,9}, {3,4,7}, {3,4,8}, {3,4,9}, {3,7,9}, {4,7,9}, {3,4,7,9}
- $\mathrm{Bi}(17)$: {3,8}, {3,11}, {4,8}, {4,11}, {8,12}, {8,13}, {11,12}, {11,13}, {3,4,8}, {3,4,11}, {3,8,11}, {3,8,12}, {3,8,13}, {3,11,12}, {3,11,13}, {4,8,11}, {4,8,12}, {4,8,13}, {4,11,12}, {4,11,13}, {8,11,12}, {8,11,13}, {8,12,13}, {11,12,13}, {3,4,8,11}, {3,4,8,12}, {3,4,8,13}, {3,4,11,12}, {3,4,11,13}, {3,8,11,12}, {3,8,11,13},

{3,8,12,13}, {3,11,12,13}, {4,8,11,12}, {4,8,11,13}, {4,8,12,13}, {4,11,12,13}, {8,11,12,13}, {3,4,8,11,12}, {3,4,8,11,13}, {3,4,8,12,13}, {3,4,11,12,13}, {3,8,11,12,13}, {4,8,11,12,13}, {3,4,8,11,12,13}

- Bi(30) : {1,9}, {1,10}, {1,11}, {3,8}, {3,9}, {3,10}, {3,11}, {4,8}, {4,9}, {4,10}, {4,11}, {9,11}, {10,11}, {1,9,11}, {1,10,11}, {3,4,8}, {3,4,9}, {3,4,10}, {3,4,11}, {3,8,11}, {3,9,11}, {3,10,11}, {4,8,11}, {4,9,11}, {4,10,11}, {3,4,8,11}, {3,4,9,11}, {3,4,10,11}

- Bi(33) : {3,7}, {3,10}, {3,12}, {3,13}, {4,7}, {4,10}, {4,12}, {4,13}, {7,11}, {7,12}, {7,15}, {10,11}, {10,13}, {10,15}, {11,12}, {11,13}, {12,15}, {13,15}, {3,4,7}, {3,4,10}, {3,4,12}, {3,4,13}, {3,7,11}, {3,7,12}, {3,7,13}, {3,7,15}, {3,10,11}, {3,10,12}, {3,10,13}, {3,10,15}, {3,11,12}, {3,11,13}, {3,12,15}, {3,13,15}, {4,7,11}, {4,7,12}, {4,7,13}, {4,7,15}, {4,10,11}, {4,10,12}, {4,10,13}, {4,10,15}, {4,11,12}, {4,11,13}, {4,12,15}, {4,13,15}, {7,11,12}, {7,11,13}, {7,11,15}, {7,12,15}, {7,13,15}, {10,11,12}, {10,11,13}, {10,11,15}, {10,12,15}, {10,13,15}, {11,12,15}, {11,13,15}, {3,4,7,11}, {3,4,7,12}, {3,4,7,13}, {3,4,7,15}, {3,4,10,11}, {3,4,10,12}, {3,4,10,13}, {3,4,10,15}, {3,4,11,12}, {3,4,11,13}, {3,4,12,15}, {3,4,13,15}, {3,7,11,12}, {3,7,11,13}, {3,7,11,15}, {3,7,12,15}, {3,7,13,15}, {3,10,11,12}, {3,10,11,13}, {3,10,11,15}, {3,10,12,15}, {3,10,13,15}, {3,11,12,15}, {3,11,13,15}, {4,7,11,12}, {4,7,11,13}, {4,7,11,15}, {4,7,12,15}, {4,7,13,15}, {4,10,11,12}, {4,10,11,13}, {4,10,11,15}, {4,10,12,15}, {4,10,13,15}, {4,11,12,15}, {4,11,13,15}, {7,11,12,15}, {7,11,13,15}, {10,11,12,15}, {10,11,13,15}, {3,4,7,11,12}, {3,4,7,11,13}, {3,4,7,11,15}, {3,4,7,12,15}, {3,4,7,13,15}, {3,4,10,11,12}, {3,4,10,11,13}, {3,4,10,11,15}, {3,4,10,12,15}, {3,4,10,13,15}, {3,4,11,12,15}, {3,4,11,13,15}, {3,7,11,12,15}, {3,7,11,13,15}, {3,10,11,12,15}, {3,10,11,13,15}, {4,7,11,12,15}, {4,7,11,13,15}, {4,10,11,12,15}, {4,10,11,13,15}, {3,4,7,11,12,15}, {3,4,7,11,13,15}, {3,4,10,11,12,15}, {3,4,10,11,13,15}

- Bi(39) : {3,9}, {3,10}, {4,9}, {4,10}, {3,4,9}, {3,4,10}, {3,9,10}, {4,9,10}, {3,4,9,10}

Appendix E. A proof of non-integrality for a Bianchi group packing

To prove non-integrality of extended Bianchi group packings, we applied Lemma 24 to solve $gV = 0$, where V is an over-determined inversive coordinate matrix of the packing's cluster and part of its orbit. In the case of non-integrality, g will have a nonlinear relation between its entries, guaranteeing a non-integral packing (see §5 for details). Below is an example, which proves non-integrality for $\widehat{Bi}(17)$ cluster $\{4, 8\}$.

$$
\begin{pmatrix}
g(1,1) & g(1,2) & g(1,3) & g(1,4) & g(1,5) & g(1,6) \\
g(2,1) & g(2,2) & g(2,3) & g(2,4) & g(2,5) & g(2,6) \\
g(3,1) & g(3,2) & g(3,3) & g(3,4) & g(3,5) & g(3,6) \\
g(4,1) & g(4,2) & g(4,3) & g(4,4) & g(4,5) & g(4,6) \\
g(5,1) & g(5,2) & g(5,3) & g(5,4) & g(5,5) & g(5,6) \\
g(6,1) & g(6,2) & g(6,3) & g(6,4) & g(6,5) & g(6,6)
\end{pmatrix}
\cdot
\begin{pmatrix}
\sqrt{17} & 0 & 0 & 1 \\
2\sqrt{34} & \sqrt{34} & \sqrt{\frac{17}{2}} & \frac{11}{\sqrt{2}} \\
\sqrt{17} & 0 & 0 & -1 \\
0 & \sqrt{17} & 0 & 1 \\
5\sqrt{17} & 4\sqrt{17} & \sqrt{17} & 18 \\
39\sqrt{17} & 16\sqrt{17} & 0 & 103
\end{pmatrix}
= 0
$$

$$
\implies
\begin{cases}
g(1,1) & \to & \frac{3g(1,2)}{\sqrt{2}} - 63g(1,6) \\
g(1,3) & \to & 24g(1,6) - \sqrt{2}g(1,2) \\
g(1,4) & \to & \sqrt{2}g(1,2) - 16g(1,6) \\
g(1,5) & \to & -\frac{g(1,2)}{\sqrt{2}} \\
g(2,1) & \to & \frac{3g(2,2)}{\sqrt{2}} - 63g(2,6) \\
g(2,3) & \to & 24g(2,6) - \sqrt{2}g(2,2) \\
g(2,4) & \to & \sqrt{2}g(2,2) - 16g(2,6) \\
g(2,5) & \to & -\frac{g(2,2)}{\sqrt{2}} \\
g(3,1) & \to & \frac{3g(3,2)}{\sqrt{2}} - 63g(3,6) \\
g(3,3) & \to & 24g(3,6) - \sqrt{2}g(3,2) \\
g(3,4) & \to & \sqrt{2}g(3,2) - 16g(3,6) \\
g(3,5) & \to & -\frac{g(3,2)}{\sqrt{2}} \\
g(4,1) & \to & \frac{3g(4,2)}{\sqrt{2}} - 63g(4,6) \\
g(4,3) & \to & 24g(4,6) - \sqrt{2}g(4,2) \\
g(4,4) & \to & \sqrt{2}g(4,2) - 16g(4,6) \\
g(4,5) & \to & -\frac{g(4,2)}{\sqrt{2}} \\
g(5,1) & \to & \frac{3g(5,2)}{\sqrt{2}} - 63g(5,6) \\
g(5,3) & \to & 24g(5,6) - \sqrt{2}g(5,2) \\
g(5,4) & \to & \sqrt{2}g(5,2) - 16g(5,6) \\
g(5,5) & \to & -\frac{g(5,2)}{\sqrt{2}} \\
g(6,1) & \to & \frac{3g(6,2)}{\sqrt{2}} - 63g(6,6) \\
g(6,3) & \to & 24g(6,6) - \sqrt{2}g(6,2) \\
g(6,4) & \to & \sqrt{2}g(6,2) - 16g(6,6) \\
g(6,5) & \to & -\frac{g(6,2)}{\sqrt{2}}
\end{cases}
$$

APPENDIX F. CONVERTING INTO INVERSIVE COORDINATES

Often, authors use alternate coordinate systems when working with $\mathbb{H}^{n+1}$ and Vinberg's algorithm. In this section, we specify the transformations used for a given

quadratic form, in order to preserve the properties of each space; namely, if it is known that for some quadratic form A, all vectors $v \in V$ have $\langle v, v \rangle_A$ with some property, then we wish to find f_A such that $\langle f_A(v), f_A(v) \rangle_Q$ satisfies an analogue to that property.

F.1. Conversion of [Mcl13]'s $\widehat{Bi}$ coordinates.

This is relevant to §5. The vectors produced by [Mcl13] were obtained using the quadratic form

$$
f = \begin{cases} -2x_1x_2 + 2x_3^2 + 2mx_4^2 & \text{if } m \equiv 1, 2 \pmod 4 \\ -2x_1x_2 + 2x_3^2 + 2x_3x_4 + \frac{m+1}{2}x_4^2 & \text{if } m \equiv 3 \pmod 4 \end{cases} \tag{54}
$$

for each $\widehat{Bi}(m)$. Our first step in obtaining extended Bianchi group packings was to convert these coordinates to coordinates that correspond to our quadratic form $Q = -1$, by which we mean that all vectors v satisfy $\langle v, v \rangle_Q = -1$ (see Definition 8 and [Kon17]). To recap, this quadratic form arose directly from Definition 7 of sphere inversion, which led to the equation

$$
\widehat{b}b - |bz|^2 = -1. \tag{55}
$$

In order to generate circle packings from $\widehat{Bi}(m)$, we converted Mcleod's coordinates to fit the 2-dimensional version of (55), $\widehat{b}b - (b\bar{x})^2 - (b\bar{y})^2$, in the following manner:*

$$
\begin{cases} (x_1, x_2, x_3, x_4\sqrt{m}) \mapsto (\widehat{b}, b, b\bar{x}, b\bar{y}) & \text{if } m \equiv 1, 2 \pmod 4 \\ (x_1, x_2, x_3 + \frac{x_4}{2}, \frac{x_4\sqrt{m}}{2}) \mapsto (\widehat{b}, b, b\bar{x}, b\bar{y}) & \text{if } m \equiv 3 \pmod 4 \end{cases} \tag{56}
$$

F.2. Conversion of [Vin72; Mcl10]'s coordinates.

This is relevant to §6. In that context, [Vin72; Mcl10] use quadratic forms $-dx_0^2 + \sum_{i=1}^{n} x_i^2$ and vectors $x = (x_i)_{i=0}^{n} \in \mathbb{R}^{n+1}$ for which $\langle x, x \rangle \in \mathbb{N}$. We apply the following conversion:

$$
f(x) = (\widehat{x}_0\sqrt{d} + \widehat{x}_1, \widehat{x}_0\sqrt{d} - \widehat{x}_1, \widehat{x}_2, \dots, \widehat{x}_n) \tag{57}
$$

where $\widehat{x} = x/\sqrt{\langle x, x \rangle}$ with components $\widehat{x}_0, \dots, \widehat{x}_n$.

Lemma 58. (57) *corresponds to valid inversive coordinates.*

---

*We converted [Mcl13]'s coordinates only after normalizing their lengths.

Proof.

$$f(x)Qf(x)^T = d\widehat{x}_0^2 - \widehat{x}_1^2 - \sum_{i=2}^{n} x_i^2$$

$$= d\left(\frac{x_0}{\sqrt{\langle x, x\rangle}}\right)^2 - \sum_{i=1}^{n}\left(\frac{x_i}{\sqrt{\langle x, x\rangle}}\right)^2$$

$$= \frac{dx_0^2 - \sum_{i=1}^{n} x_i^2}{\langle x, x\rangle}$$

$$= -\frac{\langle x, x\rangle}{\langle x, x\rangle}$$

$$= -1.$$

$\square$

Therefore, for given d, (57) gives the function used to convert to inversive coordinates, preserving the properties of the domain inner product space.

Appendix G. Known nontrivial high-dim. packings: data & proofs

In this section we present packings for quadratic forms whose Coxeter diagrams as computed from Vinberg's algorithm in [Vin72; Mcl10] do not have clusters, as mentioned in §6.

G.1. $d = 1, n = 3$. In the configuration obtained by Vinberg's algorithm in [Vin72], we double about 3 to obtain the following Coxeter diagram
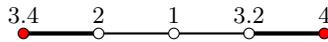
arising from the configuration

	$\widehat{b}$	b	bx	by	also equals:
1	$-\frac{\sqrt{2}}{2}$	$\frac{\sqrt{2}}{2}$	$\frac{\sqrt{2}}{2}$	0	3.1
2	0	0	$-\frac{\sqrt{2}}{2}$	$\frac{\sqrt{2}}{2}$	
4	$\sqrt{2}$	0	$\frac{\sqrt{2}}{2}$	$\frac{\sqrt{2}}{2}$	
3.2	0	0	$-\frac{\sqrt{2}}{2}$	$-\frac{\sqrt{2}}{2}$	
3.4	$\sqrt{2}$	0	$\frac{\sqrt{2}}{2}$	$-\frac{\sqrt{2}}{2}$	

(59)

which has Gram matrix

$$\begin{pmatrix} -1 & 0 & \frac{1}{2} & 0 & 1 \\ 0 & -1 & \frac{1}{2} & 1 & 0 \\ \frac{1}{2} & \frac{1}{2} & -1 & 0 & 0 \\ 0 & 1 & 0 & -1 & 0 \\ 1 & 0 & 0 & 0 & -1 \end{pmatrix}. \tag{60}$$

Interestingly, this is precisely the same as the Apollonian packing, which is also the packing for $\widehat{Bi}(1)$.

G.2. $d = 3, n = 5$. In the configuration obtained by Vinberg's algorithm in [Mcl10], we double about 5 to obtain the following Coxeter diagram
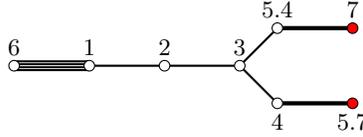
arising from the configuration

	$\widehat{b}$	b	bx_1	bx_2	bx_3	bx_4	also equals:
1	$-\frac{\sqrt{2}}{2}$	$\frac{\sqrt{2}}{2}$	$\frac{\sqrt{2}}{2}$	0	0	0	5.1
2	0	0	$-\frac{\sqrt{2}}{2}$	$\frac{\sqrt{2}}{2}$	0	0	5.2
3	0	0	0	$-\frac{\sqrt{2}}{2}$	$\frac{\sqrt{2}}{2}$	0	5.3
4	0	0	0	0	$-\frac{\sqrt{2}}{2}$	$\frac{\sqrt{2}}{2}$	
6	$\frac{\sqrt{2}+\sqrt{6}}{2}$	$\frac{\sqrt{2}-\sqrt{6}}{2}$	0	0	0	0	5.6
7	$\frac{\sqrt{2}+\sqrt{6}}{2}$	$\frac{\sqrt{6}-\sqrt{2}}{2}$	$\frac{\sqrt{2}}{2}$	$\frac{\sqrt{2}}{2}$	$\frac{\sqrt{2}}{2}$	$\frac{\sqrt{2}}{2}$	
5.4	0	0	0	0	$-\frac{\sqrt{2}}{2}$	$-\frac{\sqrt{2}}{2}$	
5.7	$\frac{\sqrt{2}+\sqrt{6}}{2}$	$\frac{\sqrt{6}-\sqrt{2}}{2}$	$\frac{\sqrt{2}}{2}$	$\frac{\sqrt{2}}{2}$	$\frac{\sqrt{2}}{2}$	$-\frac{\sqrt{2}}{2}$	

$$\tag{61}$$

which has Gram matrix

$$\begin{pmatrix} -1 & 0 & \frac{1}{2} & 0 & 0 & 0 & 0 & 1 \\ 0 & -1 & \frac{1}{2} & 0 & 0 & 0 & 1 & 0 \\ \frac{1}{2} & \frac{1}{2} & -1 & \frac{1}{2} & 0 & 0 & 0 & 0 \\ 0 & 0 & \frac{1}{2} & -1 & \frac{1}{2} & 0 & 0 & 0 \\ 0 & 0 & 0 & \frac{1}{2} & -1 & \frac{\sqrt{3}}{2} & 0 & 0 \\ 0 & 0 & 0 & 0 & \frac{\sqrt{3}}{2} & -1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & -1 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 & -1 \end{pmatrix}. \tag{62}$$

G.3. $d = 3, n = 6$.

Claim 63. *The following inversive coordinates generate a subgroup of the group of isometries obtained by Vinberg's algorithm in [Mcl10].*

	$\widehat{b}$	b	bx_1	bx_2	bx_3	bx_4	bx_5	def'd as :
9	0	0	0	0	0	0	-1	3.6
10	0	0	0	0	0	$-\frac{\sqrt2}{2}$	$\frac{\sqrt2}{2}$	3.5
11	0	0	0	$-\frac{\sqrt2}{2}$	0	$\frac{\sqrt2}{2}$	0	3.4
12	0	0	0	$-\frac{\sqrt2}{2}$	$\frac{\sqrt2}{2}$	0	0	3
13	$-\frac{\sqrt2}{2}$	$\frac{\sqrt2}{2}$	$\frac{\sqrt2}{2}$	0	0	0	0	1
14	$\sqrt2$	$\sqrt2$	0	$\frac{\sqrt6}{2}$	$\frac{\sqrt6}{2}$	0	0	$(2.1.2.7).3.(2.1).\widetilde{7}$
15	$\frac{\sqrt2+\sqrt6}{2}$	$\frac{\sqrt2-\sqrt6}{2}$	0	0	0	0	0	7
16	$\frac{\sqrt2+\sqrt6}{2}$	$\frac{\sqrt6-\sqrt2}{2}$	0	$\sqrt2$	0	0	0	2.7
17	$\frac{\sqrt2+\sqrt6}{2}$	$\frac{\sqrt6-\sqrt2}{2}$	$\frac{\sqrt2}{2}$	$\frac{\sqrt2}{2}$	$\frac{\sqrt2}{2}$	$\frac{\sqrt2}{2}$	0	3.8

$$(64)$$

This is (64)'s Gram matrix:

$$
\begin{pmatrix}
-1 & \frac{\sqrt2}{2} & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
\frac{\sqrt2}{2} & -1 & \frac{1}{2} & 0 & 0 & 0 & 0 & 0 & \frac{1}{2} \\
0 & \frac{1}{2} & -1 & -\frac{1}{2} & 0 & \frac{\sqrt3}{2} & 0 & 1 & 0 \\
0 & 0 & -\frac{1}{2} & -1 & 0 & 0 & 0 & 1 & 0 \\
0 & 0 & 0 & 0 & -1 & 0 & \frac{\sqrt3}{2} & \frac{1}{2} & 0 \\
0 & 0 & \frac{\sqrt3}{2} & 0 & 0 & -1 & 1 & 0 & 0 \\
0 & 0 & 0 & 0 & \frac{\sqrt3}{2} & 1 & -1 & 0 & 0 \\
0 & 0 & 1 & 1 & \frac{1}{2} & 0 & 0 & -1 & 0 \\
0 & \frac{1}{2} & 0 & 0 & 0 & 0 & 0 & 0 & -1
\end{pmatrix}.
$$

$$(65)$$

Lemma 66. (64) *has empty interior in* $\mathbb{R}^5$, *and extends by Poincaré extension to a hyperbolic polytope of finite volume.*

Proof. We first show that the configuration has empty interior:

$$-x_5 > 0 \tag{3.6.9}$$
$$\implies x_5 < 0$$
$$-\frac{x_4}{\sqrt{2}} + \frac{x_5}{\sqrt{2}} > 0 \tag{3.6.10}$$
$$\implies x_4 < x_5 < 0$$
$$-\frac{x_2}{\sqrt{2}} + \frac{x_4}{\sqrt{2}} > 0 \tag{3.6.11}$$
$$\implies x_2 < x_4 < 0$$

$$\left(\frac{\sqrt{6}+\sqrt{2}}{2}\right)^2 - x_1^2 - \left(\sqrt{3}+1-x_2\right)^2 - x_3^2 - x_4^2 - x_5^2 > 0 \tag{3.6.16}$$

$$\implies \left(\frac{\sqrt{6}+\sqrt{2}}{2}\right)^2 > x_1^2 + \left(\sqrt{3}+1-x_2\right)^2 + x_3^2 + x_4^2 + x_5^2 \geqslant \left(\sqrt{3}+1-x_2\right)^2$$
$$> \left(\sqrt{3}+1\right)^2$$
$$= 2\left(\frac{\sqrt{6}+\sqrt{2}}{2}\right)^2,$$

a contradiction. Hence no point $(x_i)_{i=1}^5 \in \mathbb{R}^5$ lies in the mutual interior of the specified configuration.

(3.6.16) gives bounds for each coordinate:

$$\left(\frac{\sqrt{6}-\sqrt{2}}{2}\right)^2 - x_1^2 - \left(\sqrt{3}+1-x_2\right)^2 - x_3^2 - x_4^2 - x_5^2 > 0$$

$$\implies \left(\frac{\sqrt{6}-\sqrt{2}}{2}\right)^2 > x_1^2 + \left(\sqrt{3}+1-x_2\right)^2 + x_3^2 + x_4^2 + x_5^2 \geqslant x_i^2$$

$$\implies |x_i| \leqslant \frac{\sqrt{6}-\sqrt{2}}{2}$$

for $1 \leqslant i \leqslant 5$. Since the intersection of the respective Poincaré extensions of the circles is bounded and does not meet the boundary of $\mathbb{H}^6$, it must be of finite volume. $\qquad\square$

Theorem 67. (64) *generates a sphere packing in* $\mathbb{R}^5$ *through the cluster* $\{12\}$.

Proof. Application of Theorem 4 to the following Coxeter diagram proves the result.

$\square$
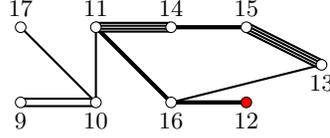
G.4. $d = 3, n = 7$.

Claim 68. *The following inversive coordinates generate a subgroup of the group of isometries obtained by Vinberg's algorithm in [Mcl10].*

	$\widehat{b}$	b	bx_1	bx_2	bx_3	bx_4	bx_5	bx_6	
10	0	0	0	0	0	0	0	-1	3.7
11	0	0	0	0	0	0	$-\frac{\sqrt{2}}{2}$	$\frac{\sqrt{2}}{2}$	3.6
12	0	0	0	0	0	$-\frac{\sqrt{2}}{2}$	$\frac{\sqrt{2}}{2}$	0	3.5
13	0	0	0	$-\frac{\sqrt{2}}{2}$	0	$\frac{\sqrt{2}}{2}$	0	0	3.4
14	0	0	0	$-\frac{\sqrt{2}}{2}$	$\frac{\sqrt{2}}{2}$	0	0	0	3
15	$-\frac{\sqrt{2}}{2}$	$\frac{\sqrt{2}}{2}$	$\frac{\sqrt{2}}{2}$	0	0	0	0	0	1
16	$\sqrt{2}$	$\sqrt{2}$	0	$\frac{\sqrt{6}}{2}$	$\frac{\sqrt{6}}{2}$	0	0	0	$(2.1.2.8).3.(2.1).\widetilde{8}$
17	$\frac{\sqrt{2}+\sqrt{6}}{2}$	$\frac{\sqrt{2}-\sqrt{6}}{2}$	0	0	0	0	0	0	8
18	$\frac{\sqrt{2}+\sqrt{6}}{2}$	$\frac{\sqrt{6}-\sqrt{2}}{2}$	0	$\sqrt{2}$	0	0	0	0	2.8
19	$\frac{\sqrt{2}+\sqrt{6}}{2}$	$\frac{\sqrt{6}-\sqrt{2}}{2}$	$\frac{\sqrt{2}}{2}$	$\frac{\sqrt{2}}{2}$	$\frac{\sqrt{2}}{2}$	$\frac{\sqrt{2}}{2}$	0	0	3.9

$$(69)$$

This is (69)'s Gram matrix:

$$
\begin{pmatrix}
-1 & \frac{\sqrt{2}}{2} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
\frac{\sqrt{2}}{2} & -1 & \frac{1}{2} & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & \frac{1}{2} & -1 & \frac{1}{2} & 0 & 0 & 0 & 0 & 0 & \frac{1}{2} \\
0 & 0 & \frac{1}{2} & -1 & -\frac{1}{2} & 0 & \frac{\sqrt{3}}{2} & 0 & 1 & 0 \\
0 & 0 & 0 & -\frac{1}{2} & -1 & 0 & 0 & 0 & 1 & 0 \\
0 & 0 & 0 & 0 & 0 & -1 & 0 & \frac{\sqrt{3}}{2} & \frac{1}{2} & 0 \\
0 & 0 & 0 & \frac{\sqrt{3}}{2} & 0 & 0 & -1 & 1 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & \frac{\sqrt{3}}{2} & 1 & -1 & 0 & 0 \\
0 & 0 & 0 & 1 & 1 & \frac{1}{2} & 0 & 0 & -1 & 0 \\
0 & 0 & \frac{1}{2} & 0 & 0 & 0 & 0 & 0 & 0 & -1
\end{pmatrix}.
$$

$$(70)$$

Lemma 71. (69) *has empty interior in $\mathbb{R}^6$, and extends by Poincaré extension to a hyperbolic polytope of finite volume.*

Proof. We first show that the configuration has empty interior:

$$-x_6 > 0 \qquad (3.7.10)$$
$$\implies x_6 < 0$$
$$-\frac{x_5}{\sqrt{2}} + \frac{x_6}{\sqrt{2}} > 0 \qquad (3.7.11)$$
$$\implies x_5 < x_6 < 0$$
$$-\frac{x_4}{\sqrt{2}} + \frac{x_5}{\sqrt{2}} > 0 \qquad (3.7.12)$$
$$\implies x_4 < x_5 < 0$$
$$-\frac{x_2}{\sqrt{2}} + \frac{x_4}{\sqrt{2}} > 0 \qquad (3.7.13)$$
$$\implies x_2 < x_4 < 0$$

$$\left(\frac{\sqrt{6}+\sqrt{2}}{2}\right)^2 - x_1^2 - \left(\sqrt{3}+1-x_2\right)^2 - x_3^2 - x_4^2 - x_5^2 - x_6^2 > 0 \qquad (3.7.18)$$

$$\implies \left(\frac{\sqrt{6}+\sqrt{2}}{2}\right)^2 > x_1^2 + \left(\sqrt{3}+1-x_2\right)^2 + x_3^2 + x_4^2 + x_5^2 + x_6^2 \geqslant \left(\sqrt{3}+1-x_2\right)^2$$

$$> \left(\sqrt{3}+1\right)^2$$

$$= 2\left(\frac{\sqrt{6}+\sqrt{2}}{2}\right)^2,$$

a contradiction. Hence no point $(x_i)_{i=1}^6 \in \mathbb{R}^6$ lies in the mutual interior of the specified configuration.

(3.7.18) gives bounds for each coordinate:

$$\left(\frac{\sqrt{6}-\sqrt{2}}{2}\right)^2 - x_1^2 - \left(\sqrt{3}+1-x_2\right)^2 - \sum_{i=3}^6 x_i^2 > 0$$

$$\implies \left(\frac{\sqrt{6}-\sqrt{2}}{2}\right)^2 > x_1^2 + \left(\sqrt{3}+1-x_2\right)^2 + \sum_{i=3}^6 x_i^2 \geqslant x_i^2$$

$$\implies |x_i| \leqslant \frac{\sqrt{6}-\sqrt{2}}{2}$$

for $1 \leqslant i \leqslant 6, i \neq 2$ and $\frac{2\sqrt{3}+2-\sqrt{6}+\sqrt{2}}{2} \leqslant x_2 \leqslant \frac{2\sqrt{3}+2+\sqrt{6}-\sqrt{2}}{2}$. Since the intersection of the respective Poincaré extensions of the circles is bounded and does not meet the boundary of $\mathbb{H}^7$, it must be of finite volume. $\qquad \square$

Theorem 72. (69) *generates a sphere packing in* $\mathbb{R}^6$ *through the cluster* $\{14\}$.

Proof. Application of Theorem 4 to the following Coxeter diagram proves the result.

$\square$

G.5. $d = 3, n = 8$.

Claim 73. *The following inversive coordinates generate a subgroup of the group of isometries obtained by Vinberg's algorithm in [Mcl10].*

	$\hat{b}$	b	bx_1	bx_2	bx_3	bx_4	bx_5	bx_6	bx_7	def'd as :
11	0	0	0	0	0	0	0	0	-1	3.8
12	0	0	0	0	0	0	0	$-\frac{\sqrt2}{2}$	$\frac{\sqrt2}{2}$	3.7
13	0	0	0	0	0	0	$-\frac{\sqrt2}{2}$	$\frac{\sqrt2}{2}$	0	3.6
14	0	0	0	0	0	$-\frac{\sqrt2}{2}$	$\frac{\sqrt2}{2}$	0	0	3.5
15	0	0	0	$-\frac{\sqrt2}{2}$	0	$\frac{\sqrt2}{2}$	0	0	0	3.4
16	0	0	0	$-\frac{\sqrt2}{2}$	$\frac{\sqrt2}{2}$	0	0	0	0	3
17	$-\frac{\sqrt2}{2}$	$\frac{\sqrt2}{2}$	$\frac{\sqrt2}{2}$	0	0	0	0	0	0	1
18	$\sqrt2$	$\sqrt2$	0	$\frac{\sqrt6}{2}$	$\frac{\sqrt6}{2}$	0	0	0	0	$(2.1.2.9).3.(2.1).\widetilde9$
19	$\frac{\sqrt2+\sqrt6}{2}$	$\frac{\sqrt2-\sqrt6}{2}$	0	0	0	0	0	0	0	9
20	$\frac{\sqrt2+\sqrt6}{2}$	$\frac{\sqrt6-\sqrt2}{2}$	0	$\sqrt2$	0	0	0	0	0	2.9
21	$\frac{\sqrt2+\sqrt6}{2}$	$\frac{\sqrt6-\sqrt2}{2}$	$\frac{\sqrt2}{2}$	$\frac{\sqrt2}{2}$	$\frac{\sqrt2}{2}$	$\frac{\sqrt2}{2}$	0	0	0	3.10

$$(74)$$

This is (74)'s Gram matrix:

$$
\begin{pmatrix}
-1 & \frac{1}{\sqrt2} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
\frac{1}{\sqrt2} & -1 & \frac{1}{2} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & \frac{1}{2} & -1 & \frac{1}{2} & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & \frac{1}{2} & -1 & \frac{1}{2} & 0 & 0 & 0 & 0 & 0 & \frac{1}{2} \\
0 & 0 & 0 & \frac{1}{2} & -1 & -\frac{1}{2} & 0 & \frac{\sqrt3}{2} & 0 & 1 & 0 \\
0 & 0 & 0 & 0 & -\frac{1}{2} & -1 & 0 & 0 & 0 & 1 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & \frac{\sqrt3}{2} & \frac{1}{2} & 0 \\
0 & 0 & 0 & 0 & \frac{\sqrt3}{2} & 0 & 0 & -1 & 1 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & \frac{\sqrt3}{2} & 1 & -1 & 0 & 0 \\
0 & 0 & 0 & 0 & 1 & 1 & \frac{1}{2} & 0 & 0 & -1 & 0 \\
0 & 0 & 0 & \frac{1}{2} & 0 & 0 & 0 & 0 & 0 & 0 & -1
\end{pmatrix}. \qquad (75)
$$

Lemma 76. (74) *has empty interior in* $\mathbb{R}^7$, *and extends by Poincaré extension to a hyperbolic polytope of finite volume.*

Proof. We first show that the configuration has empty interior:

$$-x_7 > 0 \tag{3.8.11}$$
$$\implies x_7 < 0$$

$$-\frac{x_6}{\sqrt{2}} + \frac{x_7}{\sqrt{2}} > 0 \tag{3.8.12}$$
$$\implies x_6 < x_7 < 0$$

$$-\frac{x_5}{\sqrt{2}} + \frac{x_6}{\sqrt{2}} > 0 \tag{3.8.13}$$
$$\implies x_5 < x_6 < 0$$

$$-\frac{x_4}{\sqrt{2}} + \frac{x_5}{\sqrt{2}} > 0 \tag{3.8.14}$$
$$\implies x_4 < x_5 < 0$$

$$-\frac{x_2}{\sqrt{2}} + \frac{x_4}{\sqrt{2}} > 0 \tag{3.8.15}$$
$$\implies x_2 < x_4 < 0$$

$$\left(\frac{\sqrt{6}+\sqrt{2}}{2}\right)^2 - x_1^2 - \left(\sqrt{3}+1-x_2\right)^2 - \sum_{i=3}^{7} x_i^2 > 0 \tag{3.8.20}$$

$$\implies \left(\frac{\sqrt{6}+\sqrt{2}}{2}\right)^2 > x_1^2 + \left(\sqrt{3}+1-x_2\right)^2 + \sum_{i=3}^{7} x_i^2 \geqslant \left(\sqrt{3}+1-x_2\right)^2$$

$$> \left(\sqrt{3}+1\right)^2$$

$$= 2\left(\frac{\sqrt{6}+\sqrt{2}}{2}\right)^2,$$

a contradiction. Hence no point $(x_i)_{i=1}^{7} \in \mathbb{R}^7$ lies in the mutual interior of the specified configuration.

(3.8.20) gives bounds for each coordinate:

$$\left(\frac{\sqrt{6}-\sqrt{2}}{2}\right)^2 - x_1^2 - \left(\sqrt{3}+1-x_2\right)^2 - \sum_{i=3}^{7} x_i^2 > 0$$

$$\implies \left(\frac{\sqrt{6}-\sqrt{2}}{2}\right)^2 > x_1^2 + \left(\sqrt{3}+1-x_2\right)^2 + \sum_{i=3}^{7} x_i^2 \geqslant x_i^2$$

$$\implies |x_i| \leqslant \frac{\sqrt{6}-\sqrt{2}}{2}$$

for $1 \leqslant i \leqslant 7, i \neq 2$ and $\frac{2\sqrt{3}+2-\sqrt{6}+\sqrt{2}}{2} \leqslant x_2 \leqslant \frac{2\sqrt{3}+2+\sqrt{6}-\sqrt{2}}{2}$. Since the intersection of the respective Poincaré extensions of the circles is bounded and does not meet the boundary of $\mathbb{H}^7$, it must be of finite volume. $\qquad\square$

Theorem 77. (74) *generates a sphere packing in* $\mathbb{R}^7$ *through the cluster* $\{16\}$.

Proof. Application of Theorem 4 to the following Coxeter diagram proves the result.

$\square$
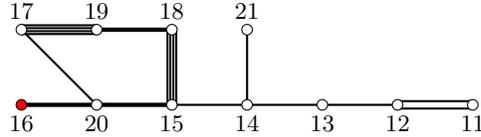
G.6. $d = 3, n = 10$.

Claim 78. *The following inversive coordinates generate a subgroup of the group of isometries obtained by Vinberg's algorithm in [Mcl10].*

	$\widehat{b}$	b	bx_1	bx_2	bx_3	bx_4	bx_5	bx_6	bx_7	bx_8	bx_9	def'd as :
15	0	0	0	0	0	0	0	0	0	0	-1	3.10
16	0	0	0	0	0	0	0	0	0	$-\frac{\sqrt2}{2}$	$\frac{\sqrt2}{2}$	3.9
17	0	0	0	0	0	0	0	0	$-\frac{\sqrt2}{2}$	$\frac{\sqrt2}{2}$	0	3.8
18	0	0	0	0	0	0	0	$-\frac{\sqrt2}{2}$	$\frac{\sqrt2}{2}$	0	0	3.7
19	0	0	0	0	0	0	$-\frac{\sqrt2}{2}$	$\frac{\sqrt2}{2}$	0	0	0	3.6
20	0	0	0	0	0	$-\frac{\sqrt2}{2}$	$\frac{\sqrt2}{2}$	0	0	0	0	3.5
21	0	0	0	$-\frac{\sqrt2}{2}$	0	$\frac{\sqrt2}{2}$	0	0	0	0	0	3.4
22	0	0	0	$-\frac{\sqrt2}{2}$	$\frac{\sqrt2}{2}$	0	0	0	0	0	0	3
23	$-\frac{\sqrt2}{2}$	$\frac{\sqrt2}{2}$	$\frac{\sqrt2}{2}$	0	0	0	0	0	0	0	0	1
24	$\sqrt2$	$\sqrt2$	0	$\frac{\sqrt6}{2}$	$\frac{\sqrt6}{2}$	0	0	0	0	0	0	*
25	$2\left(1+\sqrt3\right)$	$2\left(-1+\sqrt3\right)$	1	1	1	1	1	1	1	1	1	3.14
26	$\frac{\sqrt2+\sqrt6}{2}$	$\frac{\sqrt2-\sqrt6}{2}$	0	0	0	0	0	0	0	0	0	11
27	$\frac{\sqrt2+\sqrt6}{2}$	$\frac{\sqrt6-\sqrt6}{2}$	0	$\sqrt2$	0	0	0	0	0	0	0	2.11
28	$\frac{\sqrt2+\sqrt6}{2}$	$\frac{\sqrt6-\sqrt6}{2}$	$\frac{\sqrt2}{2}$	$\frac{\sqrt2}{2}$	$\frac{\sqrt2}{2}$	$\frac{\sqrt2}{2}$	0	0	0	0	0	3.12
29	$\frac{5\sqrt2+\sqrt6}{2}$	$\frac{5\sqrt2-\sqrt6}{2}$	$\frac{\sqrt6}{2}$	$\frac{\sqrt6}{2}$	$\frac{\sqrt6}{2}$	$\frac{\sqrt6}{2}$	$\frac{\sqrt6}{2}$	$\frac{\sqrt6}{2}$	$\frac{\sqrt6}{2}$	$\frac{\sqrt6}{2}$	0	3.13

$$(79)$$

This is (79)'s Gram matrix:

---

*(2.1.2.11).3.(2.1).$\widetilde{11}$

$$\begin{pmatrix}
-1 & \frac{\sqrt{2}}{2} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\
\frac{\sqrt{2}}{2} & -1 & \frac{1}{2} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \frac{\sqrt{3}}{2} \\
0 & \frac{1}{2} & -1 & \frac{1}{2} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & \frac{1}{2} & -1 & \frac{1}{2} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & \frac{1}{2} & -1 & \frac{1}{2} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & \frac{1}{2} & -1 & \frac{1}{2} & 0 & 0 & 0 & 0 & 0 & 0 & \frac{1}{2} & 0 \\
0 & 0 & 0 & 0 & 0 & \frac{1}{2} & -1 & -\frac{1}{2} & 0 & \frac{\sqrt{3}}{2} & 0 & 0 & 1 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & -\frac{1}{2} & -1 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & \frac{\sqrt{2}}{2} & \frac{\sqrt{3}}{2} & \frac{1}{2} & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & \frac{\sqrt{3}}{2} & 0 & 0 & -1 & \sqrt{6} & 1 & 0 & 0 & 2 \\
1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \frac{\sqrt{2}}{2} & \sqrt{6} & -1 & 0 & \sqrt{2} & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \frac{\sqrt{3}}{2} & 1 & 0 & -1 & 0 & 0 & 1 \\
0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & \frac{1}{2} & 0 & \sqrt{2} & 0 & -1 & 0 & \sqrt{3} \\
0 & 0 & 0 & 0 & 0 & \frac{1}{2} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 \\
0 & \frac{\sqrt{3}}{2} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 2 & 0 & 1 & \sqrt{3} & 0 & -1
\end{pmatrix}. \tag{80}$$

Lemma 81. (79) *has empty interior in* $\mathbb{R}^9$, *and extends by Poincaré extension to a hyperbolic polytope of finite volume.*

Proof. We first show that the configuration has empty interior:

$$-x_9 > 0 \tag{3.10.15}$$
$$\implies x_9 < 0$$

$$\left(\frac{\sqrt{6} + \sqrt{2}}{4}\right)^2 - \sum_{i=1}^{10}\left(\frac{\sqrt{6} + \sqrt{2}}{4} - x_i\right)^2 > 0 \tag{3.10.25}$$

$$\left(\frac{\sqrt{6} + \sqrt{2}}{4}\right)^2 > \sum_{i=1}^{10}\left(\frac{\sqrt{6} + \sqrt{2}}{4} - x_i\right)^2 \geqslant \left(\frac{\sqrt{6} + \sqrt{2}}{4} - x_9\right)^2,$$

$$> \left(\frac{\sqrt{6} + \sqrt{2}}{4}\right)^2, \tag{82}$$

a contradiction. Hence no point $(x_i)_{i=1}^9 \in \mathbb{R}^9$ lies in the mutual interior of the specified configuration.

(3.10.25) gives bounds for each coordinate:

$$\left(\frac{\sqrt{6}+\sqrt{2}}{4}\right)^2 - \sum_{i=1}^{10}\left(\frac{\sqrt{6}+\sqrt{2}}{4}-x_i\right)^2 > 0$$

$$\implies \left(\frac{\sqrt{6}+\sqrt{2}}{4}\right)^2 > \sum_{i=1}^{10}\left(\frac{\sqrt{6}+\sqrt{2}}{4}-x_i\right)^2 \geqslant \left(\frac{\sqrt{6}+\sqrt{2}}{4}-x_i\right)^2$$

$$\implies 0 < x_i < \frac{\sqrt{6}+\sqrt{2}}{2}$$

for $1 \leqslant i \leqslant 9$. Since the intersection of the respective Poincaré extensions of the circles is bounded and does not meet the boundary of $\mathbb{H}^{10}$, it must be of finite volume. $\quad\square$

Theorem 83. (79) *generates a sphere packing in $\mathbb{R}^9$ through the cluster $\{22\}$.*

Proof. Application of Theorem 4 to the following Coxeter diagram proves the result.

$\square$
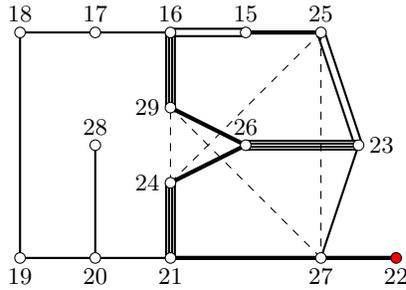
G.7. $d = 3, n = 11$.

Claim 84. *The following inversive coordinates generate a subgroup of the group of isometries obtained by Vinberg's algorithm in [Mcl10].*

	$\widehat{b}$	b	bx_1	bx_2	bx_3	bx_4	bx_5	bx_6	bx_7	bx_8	bx_9	bx_{10}	def'd as :
16	0	0	0	0	0	0	0	0	0	0	0	-1	2.3.2.11
17	0	0	0	0	0	0	0	0	0	0	$-\frac{\sqrt2}{2}$	$\frac{\sqrt2}{2}$	2.3.2.10
18	0	0	0	0	0	0	0	0	0	$-\frac{\sqrt2}{2}$	$\frac{\sqrt2}{2}$	0	2.3.2.9
19	0	0	0	0	0	0	0	0	$-\frac{\sqrt2}{2}$	$\frac{\sqrt2}{2}$	0	0	2.3.2.8
20	0	0	0	0	0	0	0	$-\frac{\sqrt2}{2}$	$\frac{\sqrt2}{2}$	0	0	0	2.3.2.7
21	0	0	0	0	0	0	$-\frac{\sqrt2}{2}$	$\frac{\sqrt2}{2}$	0	0	0	0	2.3.2.6
22	0	0	0	0	0	$-\frac{\sqrt2}{2}$	$\frac{\sqrt2}{2}$	0	0	0	0	0	2.3.2.5
23	0	0	0	$-\frac{\sqrt2}{2}$	$\frac{\sqrt2}{2}$	0	0	0	0	0	0	0	3
24	0	0	$-\frac{\sqrt2}{2}$	0	0	$\frac{\sqrt2}{2}$	0	0	0	0	0	0	2.3.2.4
25	$-\frac{\sqrt2}{2}$	$\frac{\sqrt2}{2}$	$\frac{\sqrt2}{2}$	0	0	0	0	0	0	0	0	0	1
26	$\sqrt2$	$\sqrt2$	0	$\frac{\sqrt6}{2}$	$\frac{\sqrt6}{2}$	0	0	0	0	0	0	0	*
27	$\frac{\sqrt2+\sqrt6}{2}$	$\frac{\sqrt2-\sqrt6}{2}$	0	0	0	0	0	0	0	0	0	0	12
28	$\frac{\sqrt2+\sqrt6}{2}$	$\frac{\sqrt6-\sqrt2}{2}$	0	$\sqrt2$	0	0	0	0	0	0	0	0	2.12
29	$\frac{\sqrt2+\sqrt6}{2}$	$\frac{\sqrt6-\sqrt2}{2}$	$\frac{\sqrt2}{2}$	$\frac{\sqrt2}{2}$	$\frac{\sqrt2}{2}$	$\frac{\sqrt2}{2}$	0	0	0	0	0	0	2.3.2.13
30	$\frac{5\sqrt2+\sqrt6}{2}$	$\frac{5\sqrt2-\sqrt6}{2}$	$\frac{\sqrt6}{2}$	$\frac{\sqrt6}{2}$	$\frac{\sqrt6}{2}$	$\frac{\sqrt6}{2}$	$\frac{\sqrt6}{2}$	$\frac{\sqrt6}{2}$	$\frac{\sqrt6}{2}$	$\frac{\sqrt6}{2}$	0	0	2.3.2.14
31	$\sqrt6+\sqrt2$	$\sqrt6-\sqrt2$	$\frac{\sqrt2}{2}$	$\frac{\sqrt2}{2}$	$\frac{\sqrt2}{2}$	$\frac{\sqrt2}{2}$	$\frac{\sqrt2}{2}$	$\frac{\sqrt2}{2}$	$\frac{\sqrt2}{2}$	$\frac{\sqrt2}{2}$	$\frac{\sqrt2}{2}$	$\frac{\sqrt2}{2}$	2.3.2.15

$$(85)$$

This is (85)'s Gram matrix:

---

*$(2.1.2.12).3.(2.1).\widetilde{12}$

$$\begin{pmatrix}
-1 & \frac{\sqrt{2}}{2} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \frac{\sqrt{2}}{2} \\
\frac{\sqrt{2}}{2} & -1 & \frac{1}{2} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & \frac{1}{2} & -1 & \frac{1}{2} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \frac{\sqrt{3}}{2} & 0 \\
0 & 0 & \frac{1}{2} & -1 & \frac{1}{2} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & \frac{1}{2} & -1 & \frac{1}{2} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & \frac{1}{2} & -1 & \frac{1}{2} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & \frac{1}{2} & -1 & 0 & \frac{1}{2} & 0 & 0 & 0 & 0 & \frac{1}{2} & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & \frac{1}{2} & 0 & -1 & \frac{1}{2} & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \frac{1}{2} & -1 & 0 & \frac{\sqrt{3}}{2} & \frac{1}{2} & 0 & 0 & \frac{1}{2} \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 1 & 0 & 0 & 2 & \sqrt{3} \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \frac{\sqrt{3}}{2} & 1 & -1 & 0 & 0 & 1 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & \frac{1}{2} & 0 & 0 & -1 & 0 & \sqrt{3} & 1 \\
0 & 0 & 0 & 0 & 0 & 0 & \frac{1}{2} & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 \\
0 & 0 & \frac{\sqrt{3}}{2} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 2 & 1 & \sqrt{3} & 0 & -1 & 0 \\
\frac{\sqrt{2}}{2} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \frac{1}{2} & \sqrt{3} & 0 & 1 & 0 & 0 & -1
\end{pmatrix}. \tag{86}$$

Lemma 87. (85) *has empty interior in* $\mathbb{R}^{10}$, *and extends by Poincaré extension to a hyperbolic polytope of finite volume.*

Proof. We first show that the configuration has empty interior:

$$-x_{10} > 0 \tag{3.11.16}$$
$$\implies x_{10} < 0$$
$$-\frac{x_9}{\sqrt{2}} + \frac{x_{10}}{\sqrt{2}} > 0 \tag{3.11.17}$$
$$\implies x_9 < x_{10} < 0$$
$$\left(\frac{\sqrt{6}+\sqrt{2}}{4}\right)^2 - \sum_{i=1}^{10}\left(\frac{\sqrt{3}+1}{4} - x_i\right)^2 > 0 \tag{3.11.31}$$
$$\left(\frac{\sqrt{6}+\sqrt{2}}{4}\right)^2 > \sum_{i=1}^{10}\left(\frac{\sqrt{3}+1}{4} - x_i\right)^2 \geqslant \left(\frac{\sqrt{3}+1}{4} - x_9\right)^2 + \left(\frac{\sqrt{3}+1}{4} - x_{10}\right)^2,$$
$$> 2\left(\frac{\sqrt{3}+1}{4}\right)^2,$$
$$= \left(\frac{\sqrt{6}+\sqrt{2}}{4}\right)^2,$$

a contradiction. Hence no point $(x_i)_{i=1}^{10} \in \mathbb{R}^{10}$ lies in the mutual interior of the specified configuration.

(3.11.31) gives bounds for each coordinate:

$$\left(\frac{\sqrt{6}+\sqrt{2}}{4}\right)^2 - \sum_{i=1}^{10}\left(\frac{\sqrt{3}+1}{4} - x_i\right)^2 > 0$$

$$\implies \left(\frac{\sqrt{6}+\sqrt{2}}{4}\right)^2 > \sum_{i=1}^{10}\left(\frac{\sqrt{3}+1}{4} - x_i\right)^2 \geqslant \left(\frac{\sqrt{3}+1}{4} - x_i\right)^2$$

$$\implies \left(1-\sqrt{2}\right)\frac{\sqrt{3}+1}{4} < x_i < \left(1+\sqrt{2}\right)\frac{\sqrt{3}+1}{4}$$

for $1 \leqslant i \leqslant 10$. Since the intersection of the respective Poincaré extensions of the circles is bounded and does not meet the boundary of $\mathbb{H}^{11}$, it must be of finite volume. $\square$

Theorem 88. (69) *generates a sphere packing in $\mathbb{R}^{10}$ through either of the clusters* $\{23\}$ *or* $\{26\}$.

Proof. Application of Theorem 4 to the following Coxeter diagram proves the result.

$\square$
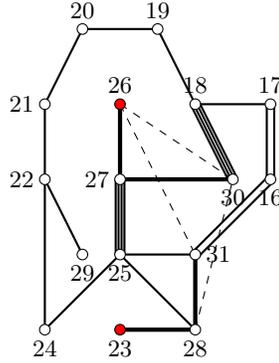
G.8. $d = 3, n = 13$.

Claim 89. *The following inversive coordinates generate a subgroup of the group of isometries obtained by Vinberg's algorithm in [Mcl10].*

	$\widehat{b}$	b	bx_1	bx_2	bx_3	bx_4	bx_5	bx_6	bx_7	bx_8	bx_9	bx_{10}	bx_{11}	bx_{12}	def'd as:
23	0	0	0	0	0	0	0	0	0	0	0	0	0	-1	2.3.2.13
24	0	0	0	0	0	0	0	0	0	0	0	0	$-\alpha$	α	2.3.2.12
25	0	0	0	0	0	0	0	0	0	0	0	$-\alpha$	α	0	2.3.2.11
26	0	0	0	0	0	0	0	0	0	0	$-\alpha$	α	0	0	2.3.2.10
27	0	0	0	0	0	0	0	0	0	$-\alpha$	α	0	0	0	2.3.2.9
28	0	0	0	0	0	0	0	0	$-\alpha$	α	0	0	0	0	2.3.2.8
29	0	0	0	0	0	0	0	$-\alpha$	α	0	0	0	0	0	2.3.2.7
30	0	0	0	0	0	0	$-\alpha$	α	0	0	0	0	0	0	2.3.2.6
31	0	0	0	0	0	$-\alpha$	α	0	0	0	0	0	0	0	2.3.2.5
32	0	0	$-\alpha$	0	0	α	0	0	0	0	0	0	0	0	2.3.2.4
33	$-\alpha$	α	α	0	0	0	0	0	0	0	0	0	0	0	1
34	$\sqrt{2}$	$\sqrt{2}$	0	γ	γ	0	0	0	0	0	0	0	0	0	*
35	$\widehat{b}_{35}$	b_{35}	0	0	0	0	0	0	0	0	0	0	0	0	14
36	$\widehat{b}_{35}$	$-b_{35}$	0	$\sqrt{2}$	0	0	0	0	0	0	0	0	0	0	2.14
37	$\widehat{b}_{35}$	b_{35}	α	α	α	α	0	0	0	0	0	0	0	0	2.3.2.15
38	$\widehat{b}_{38}$	b_{38}	γ	$\sqrt{6}$	$\sqrt{6}$	γ	γ	γ	γ	γ	γ	γ	γ	γ	2.3.2.21
39	$\widehat{b}_{39}$	b_{39}	γ	γ	γ	γ	γ	γ	γ	γ	0	0	0	0	2.3.2.16
40	$\widehat{b}_{40}$	b_{40}	$\sqrt{6}$	$\sqrt{6}$	$\sqrt{6}$	$\sqrt{6}$	$\sqrt{6}$	$\sqrt{6}$	γ	γ	γ	γ	γ	γ	2.3.2.22
41	$\widehat{b}_{41}$	b_{41}	1	1	1	1	1	1	1	1	1	1	1	1	2.3.2.20
42	$\widehat{b}_{42}$	b_{42}	$\frac{3\sqrt{2}}{2}$	$\frac{3\sqrt{2}}{2}$	$\frac{3\sqrt{2}}{2}$	$\frac{3\sqrt{2}}{2}$	$\frac{3\sqrt{2}}{2}$	$\frac{3\sqrt{2}}{2}$	$\frac{3\sqrt{2}}{2}$	α	α	α	α	α	2.3.2.19
43	$\widehat{b}_{43}$	b_{43}	α	α	$\frac{3\sqrt{2}}{2}$	α	α	α	α	α	α	α	0	0	2.3.2.17
44	$\widehat{b}_{44}$	b_{44}	α	α	$\frac{3\sqrt{2}}{2}$	α	α	α	α	α	α	α	α	α	2.3.2.18

$$(90)$$

for $\alpha = \frac{\sqrt{2}}{2}, \gamma = \frac{\sqrt{6}}{2}$, and the following values:

k	$\widehat{b}_k$	b_k
35	$\frac{\sqrt{2}+\sqrt{6}}{2}$	$\frac{\sqrt{2}-\sqrt{6}}{2}$
38	$4\sqrt{2}+\sqrt{6}$	$4\sqrt{2}-\sqrt{6}$
39	$\frac{5\sqrt{2}+\sqrt{6}}{2}$	$\frac{5\sqrt{2}-\sqrt{6}}{2}$
40	$5\sqrt{2}+\sqrt{6}$	$5\sqrt{2}-\sqrt{6}$
41	$2\sqrt{3}+1$	$2\sqrt{3}-1$
42	$\frac{5\sqrt{6}+3\sqrt{2}}{2}$	$\frac{5\sqrt{6}-3\sqrt{2}}{2}$
43	$\sqrt{6}+\sqrt{2}$	$\sqrt{6}-\sqrt{2}$
44	$\frac{3(\sqrt{6}+\sqrt{2})}{2}$	$\frac{3(\sqrt{6}-\sqrt{2})}{2}$

$$(91)$$

---

*(2.1.2.14).3.(2.1).$\widetilde{14}$

This is (90)'s Gram matrix:

$$
\begin{pmatrix}
\sharp & \alpha & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \gamma & 0 & \gamma & 1 & \alpha & 0 & \alpha \\
\alpha & \sharp & \frac{1}{2} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & \frac{1}{2} & \sharp & \frac{1}{2} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \frac{1}{2} & 0 \\
0 & 0 & \frac{1}{2} & \sharp & \frac{1}{2} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & \frac{1}{2} & \sharp & \frac{1}{2} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \beta & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & \frac{1}{2} & \sharp & \frac{1}{2} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & \frac{1}{2} & \sharp & \frac{1}{2} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \beta & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & \frac{1}{2} & \sharp & \frac{1}{2} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & \frac{1}{2} & \sharp & \frac{1}{2} & 0 & 0 & 0 & 0 & \frac{1}{2} & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \frac{1}{2} & \sharp & \frac{1}{2} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \frac{1}{2} & \sharp & 0 & \beta & \frac{1}{2} & 0 & \beta & 0 & 0 & 0 & 0 & \frac{1}{2} & 1 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \sharp & 1 & 0 & 0 & 2 & 2 & 4 & \sqrt{6} & 2\sqrt{3} & \sqrt{3} & \sqrt{3} \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \beta & 1 & \sharp & 0 & 0 & 1 & 1 & 2 & \gamma & \sqrt{3} & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \frac{1}{2} & 0 & 0 & \sharp & 0 & \sqrt{3} & \sqrt{3} & 2\sqrt{3} & \frac{3\sqrt{2}}{2} & 3 & 1 & 2 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \frac{1}{2} & 0 & 0 & 0 & 0 & 0 & \sharp & 0 & 0 & 0 & \alpha & 0 & 0 & 0 \\
\gamma & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \beta & 2 & 1 & \sqrt{3} & 0 & \sharp & 2 & 1 & 0 & \sqrt{3} & 0 & 0 \\
0 & 0 & 0 & 0 & \beta & 0 & 0 & 0 & 0 & 0 & 0 & 2 & 1 & \sqrt{3} & 0 & 2 & \sharp & 1 & \gamma & 0 & 0 & \sqrt{3} \\
\gamma & 0 & 0 & 0 & 0 & 0 & \beta & 0 & 0 & 0 & 0 & 4 & 2 & 2\sqrt{3} & 0 & 1 & 1 & \sharp & 0 & 0 & 0 & \sqrt{3} \\
1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \sqrt{6} & \gamma & \frac{3\sqrt{2}}{2} & \alpha & 0 & \gamma & 0 & \sharp & \alpha & 0 & \alpha \\
\alpha & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 2\sqrt{3} & \sqrt{3} & 3 & 0 & \sqrt{3} & 0 & 0 & \alpha & \sharp & 0 & 2 \\
0 & 0 & \frac{1}{2} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \frac{1}{2} & \sqrt{3} & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & \sharp & 0 \\
\alpha & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & \sqrt{3} & 0 & 2 & 0 & 0 & \sqrt{3} & \sqrt{3} & \alpha & 2 & 0 & \sharp
\end{pmatrix}
$$

$$(92)$$

for $\sharp = -1, \alpha = \frac{\sqrt{2}}{2}, \beta = \frac{\sqrt{3}}{2}, \gamma = \frac{\sqrt{6}}{2}$.[*]

Lemma 93. (90) *has empty interior in* $\mathbb{R}^{12}$, *and extends by Poincaré extension to a hyperbolic polytope of finite volume.*

---

[*] The use of variable names is purely due to formatting constraints due to the size of the Gram matrix.

Proof. We first show that the configuration has empty interior:

$$-x_{12} > 0 \tag{3.13.23}$$
$$\implies x_{12} < 0$$

$$\left(\frac{2\sqrt{3}+1}{7}\right)^2 - \sum_{i=1}^{12}\left(\frac{2\sqrt{3}+1}{7} - x_i\right)^2 > 0 \tag{3.11.42}$$

$$\left(\frac{2\sqrt{3}+1}{7}\right)^2 > \sum_{i=1}^{12}\left(\frac{2\sqrt{3}+1}{7} - x_i\right)^2 \geqslant \left(\frac{2\sqrt{3}+1}{7} - x_{12}\right)^2,$$

$$> \left(\frac{2\sqrt{3}+1}{7}\right)^2,$$

a contradiction. Hence no point $(x_i)_{i=1}^{12} \in \mathbb{R}^{12}$ lies in the mutual interior of the specified configuration.

(3.11.42) gives bounds for each coordinate:

$$\left(\frac{2\sqrt{3}+1}{7}\right)^2 - \sum_{i=1}^{12}\left(\frac{2\sqrt{3}+1}{7} - x_i\right)^2 > 0$$

$$\implies \left(\frac{2\sqrt{3}+1}{7}\right)^2 > \sum_{i=1}^{12}\left(\frac{2\sqrt{3}+1}{7} - x_i\right)^2 \geqslant \left(\frac{2\sqrt{3}+1}{7} - x_i\right)^2$$

$$\implies 0 < x_i < \frac{4\sqrt{3}+2}{7}$$

for $1 \leqslant i \leqslant 12$. Since the intersection of the respective Poincaré extensions of the circles is bounded and does not meet the boundary of $\mathbb{H}^{13}$, it must be of finite volume.
□

Theorem 94. (90) *generates a sphere packing in* $\mathbb{R}^{12}$ *through the cluster* $\{35\}$*.*

Proof. Application of Theorem 4 to the following Coxeter diagram proves the result.
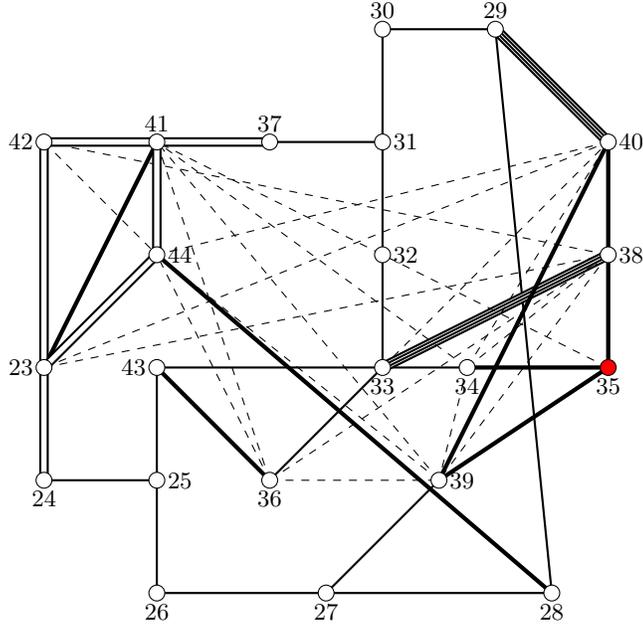
Appendix H. A note on implementing the Lobachevsky function

As in e.g. [Mil82; Vin93], we have:

Definition 95 (Lobachevsky function)**.** The *Lobachevsky function* is the integral

$$L(\theta) = \int\limits_0^\theta \log |2 \sin u| \, du. \tag{96}$$

[Mil82] discusses the importance of this function in computing exact hyperbolic volume, specifically in the case of ideal tetrahedra in $\mathbb{H}^3$, and [Vin93] provides further examples of some general computations for other hyperbolic solids. Per [Mil82], the following are also true of small θ:

$$L(\theta) = \theta \left(1 - \log |2\theta| + \sum_{n \geqslant 1} \frac{B_n (2\theta)^{2n}}{2n(n+1)!} \right) \tag{97}$$

$$L(\theta) = \frac{1}{2} \sum_{n \geqslant 1} \frac{\sin(2n\theta)}{n^2} \tag{98}$$

with (97) especially recommended for use in computation. However, comparing the runtimes of these functions using Mathematica implementations reveals that not only does the error in (97) become non-negligible for practically-sized θ, but also that in

the Mathematica architecture, (98) vastly outperforms (96) and (97) on $\theta \in [0, 2\pi)$, and that Mathematica optimizes the infinite sum to run faster than a sum with a built-in cutoff; i.e.,

$$L(\theta, N) = \frac{1}{2} \sum_{n=1}^{N} \frac{\sin(2n\theta)}{n^2} \tag{99}$$

evaluates slower than (98) even for N as small as 1000.

Acknowledgements

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