Back to Composite hypotheses.

$H_0: \theta = \theta_0$, $H_1: \theta \neq \theta_0$ / $H_1: \theta < \theta_0$

$\mathbb{P}(C \cap H_0) = \mathbb{P}_{\theta_0}(C)$. (E.g.: $H_0: \theta \geq 0.9$, $H_1: \theta < 0.9$)

More generally: $H_0: \theta \in \Theta_0$, $H_1: \theta \in \Theta_1$ $(= \Theta_0^c)$

ϑ Θ θ

Think: $\Theta_0 = \{0.9, 1\}$, $\Theta_1 = [0, 0.9)$, $\Theta_0 \cap \Theta_1 = \emptyset$.

Power function: $1 - \beta = \mathbb{P}(C; H_1)$

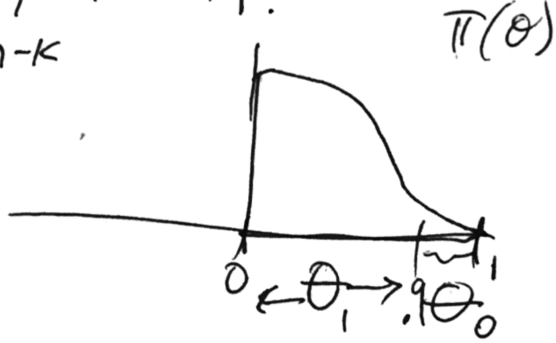
$\beta = \mathbb{P}(C^c; H_1)$

$$\pi(\theta) = \begin{cases} 1 - \beta(\theta) & \theta \in \Theta_1 \\ \alpha(\theta) & \theta \in \Theta_0 \end{cases}$$

$\alpha(\theta) = \mathbb{P}(C; \theta \in \Theta_0)$.

E.g.: $n = 20$ Bernoulli trials, $C: \{n\bar{X} \leq 14\}$.

$$\pi(\theta) = \sum_{k=0}^{14} \binom{n}{k} \theta^k (1-\theta)^{n-k}$$



OC curve: $1 - \pi(\theta)$.

Likelihood ratio statistic / test:

$$L(x_1, \dots, x_n; \theta) = f_\theta(x_1) \cdots f_\theta(x_n)$$

$$\max L_0 = \max_{\theta \in \Theta_0} L(\theta) = L(\hat{\theta}_0)$$

$$\max L = \max_{\text{all } \theta} L(\theta) = L(\hat{\theta}) \qquad \hat{\theta} = \hat{\theta}(x_1, \dots, x_n)$$

$\hat{\theta}$ ← MLE.

Λ = RV on $(X_1, \dots, X_n)$ $\qquad X_1$ takes values x_1.

$$\lambda = \frac{\max L_0(\vec{x})}{\max L(\vec{x})}$$

E.g.: $X_1, \dots, X_n$ iid $N(\mu, \sigma^2)$ — σ^2 known, μ unknown.

Test $H_0: \mu = \mu_0$ against $H_1: \mu \neq \mu_0$. $\Theta_0 = \{\mu_0\}$.

$$\max L_0 = L(\vec{x}; \underset{\hat{\mu}_0}{\mu_0}) = \frac{1}{\sqrt{2\pi\sigma^2}} e^{-\frac{(x_1-\mu_0)^2}{2\sigma^2}} \cdots \frac{1}{\sqrt{2\pi\sigma^2}} e^{-\frac{(x_n-\mu_0)^2}{2\sigma^2}}$$

$$\max L = L(\vec{x}; \overset{\bar{x}}{\hat{\mu}}) = \frac{1}{\left(\sqrt{2\pi\sigma^2}\right)^n} e^{-\frac{(x_1-\bar{x})^2}{2\sigma^2}} \cdots e^{-\frac{(x_n-\bar{x})^2}{2\sigma^2}}$$

$$\lambda = e^{-\frac{1}{2\sigma^2}\left[\sum (x_i-\mu_0)^2 - \sum (x_i - \bar{x})^2\right]}$$

$$\sum_i \left[\cancel{x_i^2} - 2x_i\mu_0 + \mu_0^2 - \cancel{x_i^2} + 2x_i\bar{x} - \bar{x}^2\right]$$

$$\to n\mu_0^2 - \cancel{n\bar{x}^2} - 2\mu_0 n\bar{x} + \cancel{2}\bar{x} n\bar{x}$$

$$\lambda = e^{-\frac{n}{2\sigma^2}\left[(\bar{x}-\mu_0)^2\right]}$$ Observation:

$$-2\log\lambda = \left(\frac{\bar{x}-\mu_0}{\sqrt{\sigma^2/n}}\right)^2 \sim \chi^2_1 \text{ deg freedom.}$$

Wilks' Thm: Assume $\circledast$ If H_0 true,

$$-2\log\Lambda \xrightarrow{(d)} \chi^2_1 \quad (\text{as } n\to\infty).$$

Sketch: $\max L_0 = L(\vec{x};\hat{\theta}_0) = f_{\hat{\theta}_0}(x_1)\cdots f_{\hat{\theta}_0}(x_n)$

$$\log L(\vec{x};\hat{\theta}_0) = \sum_i \log f_{\hat{\theta}_0}(x_i)$$

Taylor series in $\hat{\theta}_0$ near $\hat{\theta}$

$$F(\hat{\theta}_0) = F(\hat{\theta} + (\hat{\theta}_0 - \hat{\theta})).$$

$$= F(\hat{\theta}) + F'(\hat{\theta})(\hat{\theta}_0-\hat{\theta}) + \frac{F''(\hat{\theta})}{2}(\hat{\theta}_0-\hat{\theta})^2 + \text{small.}$$

$$= \underbrace{\sum_i \log f_{\hat{\theta}}(x_i)}_{\log \max L} + \cancel{\sum_i \frac{\partial}{\partial\theta}\log f_\theta(x_i)\Big|_{\theta=\hat{\theta}}(\hat{\theta}_0-\hat{\theta})} \to 0$$

$$+ \underbrace{\frac{1}{n}\sum_i \frac{\partial^2}{\partial\theta^2}\log f_\theta(x_i)\Big|_{\theta=\hat{\theta}}}_{\xrightarrow{LLN} \mathbb{E}\left(\frac{\partial^2}{\partial\theta^2}\log f_\theta\big|_{\theta=\hat{\theta}}\right) = \frac{1}{\mathrm{Var}(\hat{\theta})}} \frac{n(\hat{\theta}_0-\hat{\theta})^2}{2} + \text{small.}$$

$$\Lambda = \frac{\max L_0}{\max L} \in [0,1]$$

Likelihood Ratio Test: $C: \{\lambda < k\}$.

E.g.: $\lambda = e^{-\frac{n}{2\sigma^2}(\bar{x}-\mu_0)^2} < k$

$C: \{|\bar{x}-\mu_0| > K\}$.

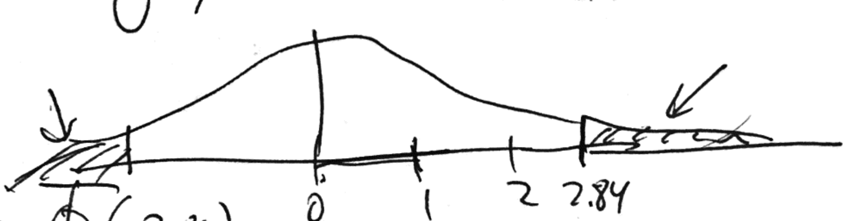
When we choose to construct confidence intervals, we make an arbitrary choice of size α. Why 95% vs 99% etc ($\alpha = 0.05$ or 0.01).

Def: P-value of a test: least α for which H_0 is rejected.

E.g.: Bakery makes 8oz packages of cookies. Pull 25 packages. Experience $\Rightarrow \sigma = 0.16$ oz. avg weight: 8.091. Test $H_0: \mu = \mu_0 = 8$ against $H_1: \mu \neq \mu_0$. (2 sided test). for significance 0.01.

Assuming H_0: $\dfrac{\bar{x}-\mu_0}{\sqrt{\sigma^2/n}} = Z = \dfrac{8.091-8}{0.16/\sqrt{25}} = 2.84$

z-value



P-value: $2\Phi(-2.84) = 0.0045$