Last time: H_0: null hypothesis

H_1: alternative hypothesis

C: critical region – where H_0 rejected.

α = size = level of significance = $\underline{\mathbb{P}(C; H_0)}$.

β = ~~power~~ = $\mathbb{P}(C^c; H_1)$, $1-\beta$ = power

L_0 = likelihood of outcome given $H_0 = \underbrace{f_{H_0}(x_1)\cdots f_{H_0}(x_n)}$.

L_1 = - - - - - - - H_1 = $\underbrace{-_{H_1} - - -_{H_1}}$.

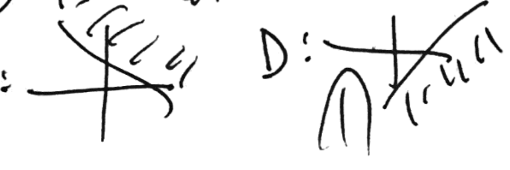
$$\alpha = \int_C L_0(\vec{x})\,\underbrace{d\vec{x}}_{dx_1\cdots dx_n}. \qquad \beta = \underbrace{\int_{C^c} L_1(\vec{x})\,d\vec{x}}.$$

Neyman-Pearson Lemma: Let C be a critical region of size α ~~& ...~~ in testing $\underline{H_0: \theta = \theta_0}$ against $\underline{H_1: \theta = \theta_1}$ & Assume there exists $\exists\, k > 0$: ① $\forall \vec{x} \in C$, $L_0(\vec{x}) \le k L_1(\vec{x})$.

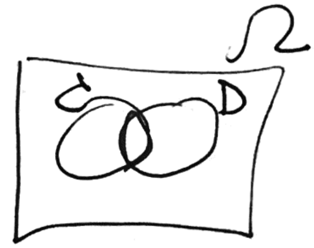
② $\forall \vec{x} \in C^c$, $L_0(\vec{x}) \ge k L_1(\vec{x})$. Then C is a **most powerful** region of size α for this test.

i.e. If D is another crit region of size α, $\int_{D^c} L_1 \ge \int_{C^c} L_1$.

C: D:

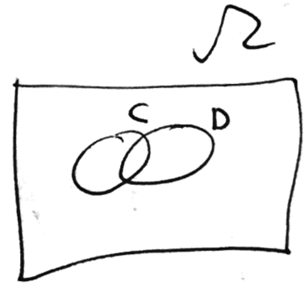


pf.:



$$\int_{D\cap C} + \int_{D\cap C^c} = \int_D L_0 = \alpha = \int_C L_0 = \int_{C\cap D} + \int_{C\cap D^c}.$$

$$\Rightarrow \int_{D\cap C^c} kL_1 \le \int_{D\cap C^c} L_0 = \int_{C\cap D^c} L_0 \le \int_{C\cap D^c} kL_1$$



$$\Rightarrow \underbrace{\int_{D^c\cap C^c} L_1 + \int_{D\cap C^c} L_1}_{\int_{C^c} L_1} \le \int_{C\cap D^c} L_1 + \int_{D^c\cap C^c} L_1$$

$$\int_{C^c} L_1 \le \int_{D^c} L_1$$

What about <u>composite</u> hypotheses?

<u>E.g.:</u> $n=20$, $C: k\le 14$, $H_0: \underline{\theta \ge 0.9}$, H_1: ~~$\theta = 0.6$~~ $\theta < 0.9$.

No longer have L_0 for all of H_0, but we still can compute $L_0(\theta)$ $\forall\theta\in H_0$.

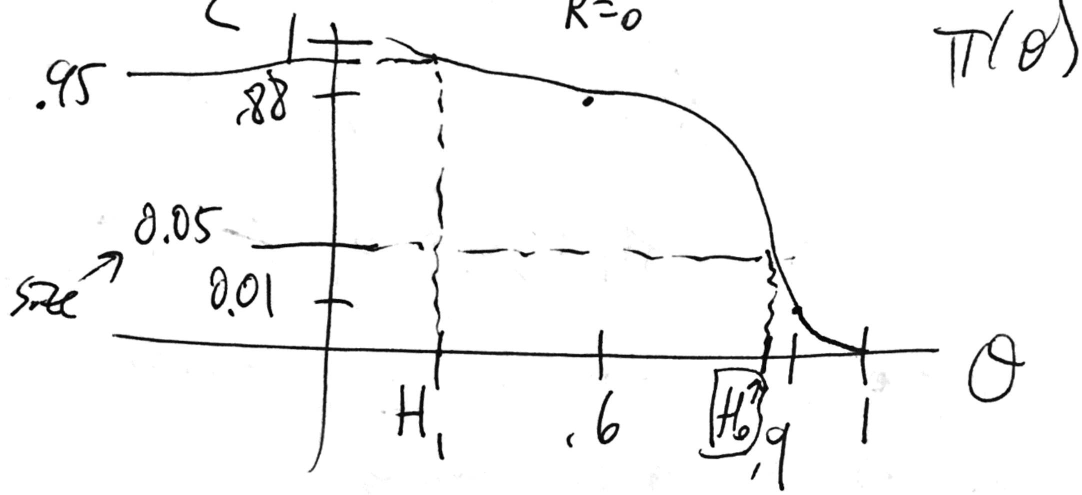
$\uparrow$ for all

<u>Def:</u> The power function $\Pi(\theta)$ of a test of H_0 vs H_1 with crit region C:

$$\Pi(\theta) = \begin{cases} \alpha(\theta) & \theta \in H_0 \\ 1-\beta(\theta) & \theta \in H_1 \end{cases}$$

<u>E.g.:</u> For $\theta \in H_0$, i.e. $\theta \geq .9$, $\alpha(\theta) = \mathbb{P}(\text{Type I error})$

$$= \int_C L_0(\theta) = \sum_{k=0}^{14} \binom{20}{k} \theta^k (1-\theta)^{20-k}$$



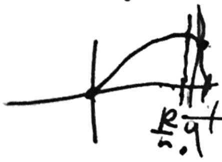
For $\theta \in H_1$, i.e. $\theta < .9$ $\beta(\theta) = \mathbb{P}(\text{Type II error})$

$$\boxed{1-\beta(\theta) = \mathbb{P}(\text{no type II error}) = \int_C L_1(\theta)} \qquad = \int_{C^c} L_1(\theta)$$

$$1-\beta(\theta) = \sum_{k=0}^{14} \binom{20}{k} \theta^k (1-\theta)^{20-k}$$

<u>Def:</u> $1-\Pi(\theta)$: the OC-curve (operating characteristic)

$L_0(\theta)$ = likelihood function given $X_1 = x_1, \ldots, X_n = x_n$,
assuming θ = parameter size, $\theta \in H_0$.



If $\theta \in H_0$, $\underline{L_0(\theta) = \binom{20}{k}\theta^k(1-\theta)^{n-k} \quad (n=20).}$

$\max L(\theta)$
over all possible values of θ

For what value of θ is this maximal?

$$\hat{\theta} = \frac{k}{n}.$$

$$\max L(\theta) = L(\hat{\theta}) = \binom{20}{k}\left(\frac{k}{n}\right)^k\left(\frac{n-k}{n}\right)^{n-k} \quad (n=20).$$

$$\max_{\theta \in H_0 \Leftrightarrow \theta \geq 0.9} L_0(\theta) = \begin{cases} L(\hat{\theta}). & \text{if } \hat{\theta} = \frac{k}{n} \geq .9. \\ L(.9). & \text{if } \hat{\theta} = \frac{k}{n} < .9. \end{cases}$$

Def: likelihood ratio statistic

$$\lambda = \frac{\max_{\theta \in H_0} L_0(\theta)}{\max_{\text{all } \theta} L(\theta)}$$