Last time: Vocabulary:

Hypothesis: proposition that is true/false

H_0 = null hypothesis ⟸ want to reject.

H_1 = alternative hypothesis ⟸ guilty

$C \subset \Omega$ outcomes is the critical region where H_0 rejected

$\alpha = \mathbb{P}(\text{Type I}) =$ "size" = level of significance.

$= \mathbb{P}(H_0 \text{ rejected but true}) = \mathbb{P}(C ; H_0)$.

$\beta = \mathbb{P}(\text{Type II}) = \mathbb{P}(H_0 \text{ accepted } \& \text{ } \cancel{H_0 \text{ false}}\ H_1) = \mathbb{P}(C^c ; H_1)$.

$\boxed{1-\beta = \text{"power of test"}}$.

E.g.: $H_0: \theta = .9$ $\qquad C: \{ n\bar{X} < 14 \}$. out of $n=20$.

$H_1: \theta = .6$

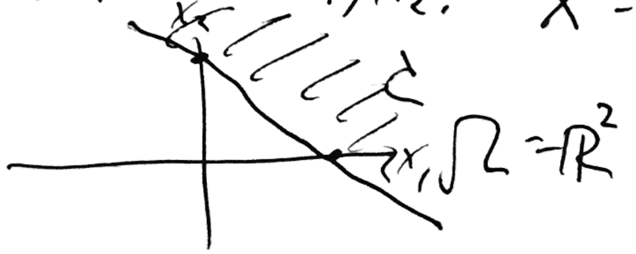
Def: simple vs composite hypothesis

simple: completely determines pdf sample. composite: not.

E.g.: X_i iid normal, $H_0: \mu = \mu_0$ composite.

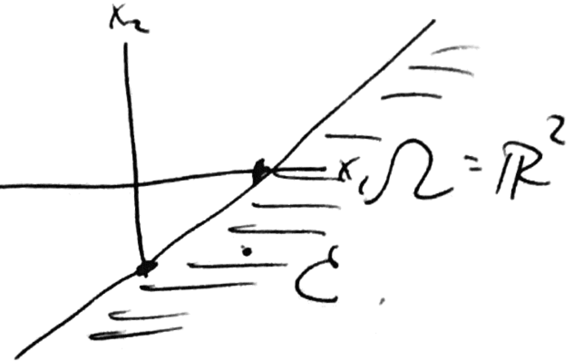
Want to test H_0 against H_1. E.g.: want estimator for μ from X_1, X_2. $\bar{X} = \frac{1}{2}(X_1 + X_2)$.

$C = \{ \bar{X} > K \}$



$\partial C = \{ x_1 + x_2 = 2K \}$

$C_1 = \{ 2X_1 - X_2 > K \}$.



Last time: chose K so that
$size = \alpha = \mathbb{P}(C_1 \mid H_0)$.
$= \mathbb{P}(C \,\&\, H_0)$.

Which of these regions is <u>most powerful</u> for H_1

<u>Recall</u>: likelihood $L(x_1, \dots, x_n ; H_0) = L_0 =$

$$L_0 = f_{X_i ; H_0}(x_1) \cdots f_{X_i ; H_0}(x_n)$$

<u>E.g.</u>: H_0: X_i are $N(\mu_0, 1)$.

$$L_0(X_1 = x_1, \dots, X_n = x_n) = \frac{1}{\sqrt{2\pi}} e^{-\frac{(x_1 - \mu_0)^2}{2}} \cdots \frac{1}{\sqrt{2\pi}} e^{-\frac{(x_n - \mu_0)^2}{2}}$$

H_1: X_i are $N(\mu_1, 1)$, $L_1 = \text{- - -} \mu_1 \text{- - -} \mu_1$.

<u>Neyman-Pearson Lemma</u>: Suppose C has size α & is such that: $\Omega \setminus C = C^c$.

① $\dfrac{L_0(\vec{x})}{L_1(\vec{x})} \leq k \quad \forall x \in C$. & ② $\dfrac{L_0(\vec{x})}{L_1(\vec{x})} \geq k \quad \forall x \in C^c$.

Then C is a <u>most powerful</u> critical region for H_0 vs H_1

most powerful critical region.

→ <u>E.g.</u>: Construct a most powerful critical region.

Look at $\frac{L_0}{L_1} = \frac{e^{-\frac{(x_1-\mu_0)^2}{2}} \cdots e^{-\frac{(x_n-\mu_0)^2}{2}}}{e^{-\frac{(x_1-\mu_1)^2}{2}} \cdots e^{-\frac{(x_n-\mu_1)^2}{2}}}$

$$= \frac{e^{-\frac{1}{2}\left[\cancel{x_1^2} - 2x_1\mu_0 + \mu_0^2 + \cancel{x_2^2} - 2x_2\mu_0 + \mu_0^2 + \cdots + \cancel{x_n^2} - 2x_n\mu_0 + \mu_0^2\right]}}{e^{-\frac{1}{2}\left[\cancel{x_1^2} - 2x_1\mu_1 + \mu_1^2 + \cancel{x_2^2} - 2x_2\mu_1 + \mu_1^2 + \cdots + \cancel{x_n^2} - 2x_n\mu_1 + \mu_1^2\right]}}$$

$$= e^{+\cancel{\frac{1}{2}}\left[+\cancel{2}x_1\mu_0 - \cancel{2}x_1\mu_1 + \cancel{2}x_2(\mu_0-\mu_1) \cdots + \cancel{2}x_n(\mu_0-\mu_1)\right]}$$

$$\cdot\, e^{-\frac{1}{2}\left[n\mu_0^2 - n\mu_1^2\right]}$$

$$= \left[e^{-\frac{1}{2}\{n(\mu_0^2-\mu_1^2)\}}\right] e^{(\mu_0-\mu_1)\left[x_1+x_2+\cdots+x_n\right]}$$

$$= \left[\underbrace{(*)\,(\text{no } x_i\text{'s})}_{\text{just constant}}\right] e^{(\mu_0-\mu_1)\cdot n\cdot\bar{x}}$$

So $\frac{L_0}{L_1}$ is a function of $\bar{x}$ so if $\frac{L_0}{L_1} \le k$ inside C

& $\frac{L_0}{L_1} \ge k$ outside C,

→ then $C: \{\bar{x} > K\}$.

pf of Lemma: Suppose C is such that

$\frac{L_0}{L_1} \le k$ in C & $\frac{L_0}{L_1} \geq k$ in C^c. Let D be another crit. region of size α. i.e.

$$\alpha = \int_C L_0(\vec{x})\,d\vec{x} = \int_D L_0(\vec{x})\,dx$$

$$= \int_{\underline{C\cap D}} L_0 + \int_{C\cap D^c} L_0 = \int_{\underline{D\cap C}} L_0 + \int_{D\cap C^c} L_0$$

$$\Rightarrow \int_{C\cap D^c} L_0 = \int_{D\cap C^c} L_0 \le k \int_{C\cap D^c} L_1$$

inside C, $L_0 \le kL_1$

outside C so $L_0 \geq kL_1$

$$\geq k\int_{D\cap C^c} L_1$$

$C = D$

$C\cap D^c$ $\int L_1$ $D\cap C^c$ $\int L_1$

$$\Rightarrow \cancel{k}\int_{D\cap C^c} L_1 + \int_{D\cap C} L_1 \le \cancel{k}\int_{C\cap D^c} L_1 + \int_{D\cap C} L_1$$

$$\int_D L_1 \le \int_C L_1$$

(#)

To finish the proof, note that the power of the test when using critical region C is: $1-\beta = 1 - \mathbb{P}(\text{Type II error})$

$$= 1 - \mathbb{P}(C^c; H_1) = \mathbb{P}(C; H_1)$$

$$= \int_C L_1(\vec{x})\, d\vec{x}.$$

So the last inequality,

$$\int_D L_1(\vec{x})\, d\vec{x} \leq \int_C L_1(\vec{x})\, dx,$$

means that the power of C is greater than that of D. Since D was <u>any</u> critical region of size α, this implies that C is indeed a <u>most powerful</u> critical region for testing H_0 against H_1. (I was trying to say this in terms of β, but its just as well to prove the inequality with $1-\beta$...)