Cramer-Rao $\hat{\theta} = \hat{\theta}(X_1, \ldots, X_n)$, iid f.

unbiased estimator for θ. $f(x;\vec{\theta}) = f(x)$.

E.g.: $f(x) = e^{-\frac{(x-\mu)^2}{2\sigma^2}} \cdot \frac{1}{\sqrt{2\pi\sigma^2}}$ $\qquad \frac{\partial}{\partial\mu} f = f \cdot \left(\frac{+2(x-\mu)}{2\sigma^2}\right)$

$\mathbb{E}(\hat{\theta}) = \theta$. E.g.: $\mu = \theta$

Thm: $\boxed{\text{Assume: } \frac{\partial}{\partial\theta}\int f(x)\,dx = \int \frac{\partial}{\partial\theta} f(x)\,dx}$

(Fails: if, e.g. $f(x) = \begin{cases} \frac{1}{\beta} & 0<x<\beta \\ 0 \end{cases}$). Then

$$\mathrm{Var}(\hat{\theta}) \geq \frac{1}{n\,\mathbb{E}\left(\left(\frac{\partial}{\partial\theta}\log f\right)^2\right)}$$

Warmup: $1 = \mathbb{E}(1) = \int 1 \cdot f(x)\,dx$. Apply $\frac{\partial}{\partial\theta}$.

$$0 = \int \frac{\partial}{\partial\theta} f(x)\,dx = \int \frac{\frac{\partial}{\partial\theta} f(x)}{f(x)} \cdot f(x)\,dx.$$

$$\boxed{0 = \mathbb{E}\left(\frac{\partial}{\partial\theta}(\log f)\right)} \qquad \frac{\partial}{\partial\theta}(\log f).$$

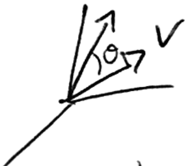
E.g.: $f(x) = e^{-\frac{(x-\mu)^2}{2\sigma^2}} \frac{1}{\sqrt{2\pi\sigma^2}}$, $\log f = \left(-\frac{(x-\mu)^2}{2\sigma^2}\right) - \log\left(\sqrt{2\pi\sigma^2}\right)$

Take $\theta = \mu$, $\frac{\partial}{\partial\theta} \log f = \frac{-2(x-\mu)}{2\sigma^2}(-1)$

$$\mathbb{E}\left(\frac{+2(x-\mu)}{2\sigma^2}\right) = \frac{1}{\sigma^2}\mathbb{E}(x-\mu) = 0.$$

Aside: Cauchy-Schwarz inequality:

$v, w \in V/\mathbb{R}$, $\langle \cdot, \cdot \rangle$, $V = \mathbb{R}^n$, $\langle (v_1, \ldots, v_n), (w_1, \ldots, w_n) \rangle = v_1 w_1 + v_2 w_2 + \ldots + v_n w_n$.



$\mathbb{R}^n$ $|\langle v, w \rangle| = \|v\| \cdot \|w\| |\cos\theta| \leq \|v\| \cdot \|w\|$.

where $\|v\| = \langle v, v \rangle^{1/2}$.

$f, g : \mathbb{R} \to \mathbb{R}$, $\int_{\mathbb{R}} f(x) g(x) dx = \langle f, g \rangle$.

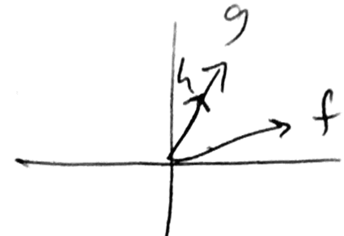
$(V = L^2 \cap L^1)$.

(C-S) Thm: $\langle f, g \rangle^2 \leq \langle f, f \rangle \cdot \langle g, g \rangle$.

i.e. $\boxed{\left(\int f(x) g(x) dx\right)^2 \leq \left(\int |f|^2\right) \cdot \left(\int |g|^2\right)}$.

(Hölder).

Pf: let $h = g - \frac{\langle g,f\rangle}{\langle f,f\rangle}\cdot f$



$$0 \le \langle h,h\rangle = \left\langle g - \frac{\langle g,f\rangle}{\langle f,f\rangle}\cdot f,\ g - \frac{\langle g,f\rangle}{\langle f,f\rangle}\cdot f \right\rangle$$

$$= \langle g,g\rangle \underbrace{- 2\left\langle g, \frac{\langle g,f\rangle}{\langle f,f\rangle}\cdot f\right\rangle}_{-2\frac{\langle g,f\rangle}{\langle f,f\rangle}\cdot\langle g,f\rangle} + \left(\frac{\langle g,f\rangle}{\langle f,f\rangle^{\cancel{2}}}\right)^{2} \cancel{\langle f,f\rangle}$$

$$0 \le \langle g,g\rangle - \frac{\langle g,f\rangle^2}{\langle f,f\rangle} \Rightarrow \frac{\langle g,f\rangle^2}{\langle f,f\rangle} \le \langle g,g\rangle.$$

$$\underline{\mathrm{Var}(\hat\theta)} = \int\cdots\int\left(\hat\theta(x_1,\dots,x_n) - \theta\right)^2 f(x_1)\cdots f(x_n)\,dx_1\cdots dx_n.$$

$$\text{''}$$

$$\mathbb{E}\left(\left(\hat\theta - \mathbb{E}(\hat\theta)\right)^2\right)$$

$$0 = \mathbb{E}(\hat\theta - \theta) = \int\cdots\int \underbrace{\left(\hat\theta(x_1,\dots,x_n) - \theta\right) f(x_1)\cdots f(x_n)}_{f} dx_1\cdots dx_n.$$

Apply $\frac{\partial}{\partial\theta}$. $\Rightarrow 0 = \int\cdots\int \underbrace{\frac{\partial}{\partial\theta}\left[\hat\theta - \theta\right]} f(x_1)\cdots f(x_n)\,d\vec{x}$

$$+ \int\cdots\int(\hat\theta - \theta)\,\frac{\partial}{\partial\theta}\left[f(x_1)\cdots f(x_n)\right]d\vec{x}$$

$\frac{\partial}{\partial\theta}\int f = \int\frac{\partial}{\partial\theta} f$

$\boxed{\partial[FG] = \partial F\, G + F\partial G}$

$$\frac{\partial}{\partial\theta}\left[\hat\theta - \theta\right] = 0 - 1 = -1.$$

$$0 = -1 \cdot \underbrace{\cancel{\int\cdots\int f(x_1)\cdots f(x_n)\, d\vec{x}}}_{\to 1}$$

$$+ \int\cdots\int (\hat{\theta}-\theta)\underbrace{\left[\frac{\partial}{\partial\theta}f_1(x_1)\, f_2 \cdots f(x_n) + f_1(x_1)\frac{\partial f}{\partial\theta}(x_2)\cdot f(x_3)\cdots f(x_n) + \cdots + f(x_1)\cdots f(x_{n-1})\frac{\partial}{\partial\theta}f(x_n)\right]}_{\underbrace{f(x_1)\cdots f(x_n)}\underbrace{\left[\frac{\frac{\partial}{\partial\theta}f(x_1)}{f(x_1)} + \frac{\frac{\partial}{\partial\theta}f(x_2)}{f(x_2)} + \cdots + \frac{\frac{\partial}{\partial\theta}f(x_n)}{f(x_n)}\right]}_{\left[\frac{\partial}{\partial\theta}\left(\log f(x_1)\right) + \cdots + \frac{\partial}{\partial\theta}\log f(x_n)\right]}} d\vec{x}.$$

$$\Rightarrow 1^2 = \left[\int\cdots\int \underbrace{(\hat{\theta}-\theta)\left(f(x_1)\cdots f(x_n)\right)^{1/2}}\ \underbrace{\left[\frac{\partial}{\partial\theta}\log f(x_1) + \cdots + \frac{\partial}{\partial\theta}\log f(x_n)\right]\left(f(x_1)\cdots f(x_n)\right)^{1/2}} d\vec{x}\right]^2$$

$\boxed{C\text{-}S}$

$$1 \le \underbrace{\left(\int\cdots\int (\hat{\theta}-\theta)^2 f(x_1)\cdots f(x_n)\, d\vec{x}\right)}_{Var(\hat{\theta})} \cdot \underbrace{\left(\int\cdots\int \left[\frac{\partial}{\partial\theta}\log f(x_1) + \cdots + \frac{\partial}{\partial\theta}\log f(x_n)\right]^2 f(x_1)\cdots f(x_n)\, d\vec{x}\right)}_{\int\cdots\int\left[\left(\frac{\partial}{\partial\theta}\log f(x_1)\right)^2 + \cdots + \left(\frac{\partial}{\partial\theta}\log f(x_n)\right)^2 + \sum_{i\neq j}\left(\frac{\partial}{\partial\theta}\log f(x_i)\cdot\frac{\partial}{\partial\theta}f(x_j)\right)\right] f(x_1)\cdots f(x_n)\, d\vec{x}}$$

$$= Var(\hat{\theta})\left[\underbrace{n\,\mathbb{E}\left(\frac{\partial}{\partial\theta}\log f\right)^2} + \underbrace{\cancel{\sum_{i\neq j}\mathbb{E}\left(\frac{\partial}{\partial\theta}\log f(x_i)\cdot\frac{\partial}{\partial\theta}\log f(x_j)\right)}}_{\cancel{\mathbb{E}\left(\frac{\partial}{\partial\theta}\log f\right)\cdot\mathbb{E}\left(\frac{\partial}{\partial\theta}\log f\right)}}\right].$$

Exercise 1: Verify for Gaussian that

$\hat{\mu} = X\hat{\beta}$ optimal for μ.

Exercise 2: Compare $n\,\mathbb{E}\left(\left(\frac{\partial}{\partial\theta}\log f\right)^2\right)$ for

Uniform on $[0, \beta]$ with

$\underbrace{\frac{(n+1)}{n} Y_n}$
