DIMACS: Probability Week- Day 2 Instructor: Abigail Raz- ajr224@math.rutgers.edu Summer 2017

Name:

1 Conditional Probability

Additional information can change our probabilities.

Definition 1. If A and B are events with $\mathbb{P}(B) \neq 0$, the conditional probability of A given B is

$$\mathbb{P}(A|B) = \frac{\mathbb{P}(A \cap B)}{\mathbb{P}(B)}$$

Example 1. A room of people has 30 people in it. There are 15 soccer fans and 26 baseball fans. There are 11 people that like both. We pick a person uniformly at random (all outcomes equally likely.) Let S be the event the person is a soccer fan and B the event the person is a baseball fan.

- 1. What is $\mathbb{P}(S|B)$? What is $\mathbb{P}(S^c|B)$?
- 2. What is the conditional probability that they like baseball given that they are a soccer fan?

Example 2. Alice and Bob are playing bridge (they are each dealt a 13 card hand). Let A be the event that Alice has no diamonds in her hand, and let B be the event that Bob has no diamonds in his hand. Try to figure out $\mathbb{P}(A \cap B)$ in as many different ways as possible.

Example 3. A gambler has two coins. One is fair and the other has both sides as heads. The gambler picks each coin with equal probability. Let R be the event that the real coin is selected.

- 1. Suppose the gambler flips the selected coin and it comes up heads. What is the probability of R given this information?
- 2. Suppose the gambler flips the selected coin N times and it comes up heads every time. What is the probability of R given this information?
- 3. Suppose the gambler flips the selected coin twice and gets heads both times, but then flips a third time and gets tails. What is the probability of R given this information?

Theorem 1. (Law of total probability) Let A be an event and $B_1, B_2, \ldots B_n$ be events that partition the sample space. Then

$$\mathbb{P}(A) = \sum_{i=1}^{n} \mathbb{P}(B_i) \mathbb{P}(A|B_i)$$

Proof:

Theorem 2. Let A be an event where $\mathbb{P}(A) > 0$. For any event $E \subseteq S$, define $\mu(E) = \mathbb{P}(E|A)$. Then μ satisfies the axioms of a probability measure.

[Proof on homework]

Example 4. (Monty Hall Problem). There are three doors. Behind one is a new car, while the other two have goats behind them. The contestant picks a door. Then the host (Monty Hall) opens a different door to reveal a goat. (If both the unpicked doors have goats then Monty picks which to open at random). After the reveal the contestant is given the option to switch the door they picked.

- Should the contestant switch doors?
- Suppose there are now 100 doors with a car behind only one. The host will now open up 98 of the unpicked doors to reveal all goats. Should this contestant switch doors?

Example 5. An ant is walking along the edges of a cube. Each time it gets to a corner it picks one of the three edges at random to walk along (each edge equally likely). Suppose the ant starts at corner A. What is the probability that the ant reaches the opposite corner of the cube before returning to A again?

Definition 2. Two events A and B are independent if and only if $\mathbb{P}(A \cap B) = \mathbb{P}(A)\mathbb{P}(B)$. In general n events A_1, \ldots, A_n are independent if and only if $P(A_1 \cap A_2 \cdots \cap A_n) = \mathbb{P}(A_1)\mathbb{P}(A_2) \cdots \mathbb{P}(A_n)$ AND every collection of n-1 events is independent.

Example 6. Two fair dice are rolled. Let E be the event that the first dice shows a 5. Determine whether E is independent of the following events and if not determine whether the probability of the intersection or the product of the probabilities is bigger.

- 1. F_1 : The sum of the two dice is 8.
- 2. F_2 : The second dice is a 4.
- 3. F_3 : The sum of the two dice is a 3.

Example 7. There are 40 people in a room each with a fair coin. They all start standing up, and independently flip their coin. The people who get heads must sit down. What is the probability that there will be at least one person standing after 7 flips?

Exercises

- 1. Prove the following theorems:
 - (a) Let A be an event where $\mathbb{P}(A) > 0$. For any event $E \subseteq S$ define $\mu(E) = \mathbb{P}(E|A)$. Then μ is a probability measure.
 - (b) If the events A and B are independent, then the events A and B^c must be independent as well. Moreover, the events A^c and B^c are independent.
- 2. For each statement below, either prove it if it is true or provide a counter-example if it is false.
 - (a) If A and B are independent with $\mathbb{P}(B) \neq 0$, then $\mathbb{P}(A|B) = \mathbb{P}(A)$
 - (b) It is impossible for two events to be both independent and disjoint.
 - (c) Suppose $0 < \mathbb{P}(B) < 1$. If $\mathbb{P}(A|B) > \mathbb{P}(A)$ then $\mathbb{P}(A|B^c) < \mathbb{P}(A)$.
 - (d) Suppose $0 < \mathbb{P}(B) < 1$. If $\mathbb{P}(A|B) > \mathbb{P}(A)$ then $\mathbb{P}(A^c|B) > \mathbb{P}(A^c)$.
- 3. (a) Find an example of three events, A, B, and C, such that any two of them are independent but $\mathbb{P}(A \cap B \cap C) \neq \mathbb{P}(A)\mathbb{P}(B)\mathbb{P}(C).$
 - (b) Generalize this.
- 4. (Gambler's Ruin) A gambler plays a game many times in a row. He starts with \$3. Each time he plays he has an equal chance or winning or losing and every round is independent. If he wins he gains \$1 otherwise he loses \$1. The gambler continues to play until he has \$10 or has run out of money.
 - (a) What is the probability that he will lose all his money before earning \$10.
 - (b) Generalize this result.
- 5. (Euler's φ function) In number theory, the function $\varphi(n)$ is the number of integers x such that $1 \le x \le n$ and gcd(x, n) = 1. For example, $\varphi(6) = 2$ since 1 and 5 are the only such numbers.
 - (a) Compute $\varphi(n)$ for $1 \le n \le 12$.
 - (b) Suppose we pick Z uniformly at random from 1 to n. Let G be the event that gcd(Z, n) = 1. Write $\varphi(n)$ in terms of $\mathbb{P}(G)$.
 - (c) For an integer a let D_a be the event that a does not divide Z. If a does divide n what is $\mathbb{P}(D_a)$?
 - (d) Suppose a and b are integers such that ab divides n and gcd(a, b) = 1. Show that D_a and D_b are independent.
 - (e) Assume $p_1, \ldots p_k$ are all the distinct prime factors of n. Write the event G in terms of D_{p_i} , and use this to find a formula for $\varphi(n)$.
- 6. (Penney's Game)Two players decide to play a coin flipping game. There are five possible patterns that each player is allowed to choose from. The patterns are

$\{HHT, THH, TTH, HTT, HTH\}.$

Each player picks one of these patterns, and they aren't allowed to pick the same one. Then the game begins, and a fair coin is flipped over and over with each flip independent of the others. The

person whose pattern first appears as three consecutive flips is the winner. For example: Suppose player A picks the pattern HHT and player B picks the pattern THH. Suppose the coin flips are HTHTTHTTHH. The game would end on the last flip since this is the first time that either player's pattern showed up as three flips in a row. This pattern is THH, so player B would be the winner.

- (a) For each possible pair of patterns, P1 and P2, determine the probability that pattern P1 appears before pattern P2. (Hint: condition on the first flip.)
- (b) If you were playing this game, would you rather select your pattern first or second? Describe your strategy.