Joint Distributions, Moment Generating Functions, Central Limit Theorem

1. A fair coins is flipped twice. Let X be 1 if the first coin flip lands heads, and 0 if it lands tails. Let Y be the total number of heads. Find the joint mass function of X and Y. Find the marginal mass functions of X and Y. Are X and Y independent?

2. Two fair dice are rolled. Find the joint probability mass function of X and Y given that they are the following. Find the marginal probability mass functions of X and Y.

   (a) X is the largest value obtained on any die, and Y is the sum of the two values,

   (b) X is the value on the first die, and Y is the larger of the two values.

   (c) X is the smallest, and Y is the largest value obtained on the dice.

3. Let X and Y have the joint density function

$$f(x,y) = \begin{cases} x + \frac{1}{2}y & 0 \le x \le 2y \le 2 \\ 0 & \text{otherwise.} \end{cases}$$

   (a) Verify that the marginal densities, for X and Y respectively, are

$$f_X(x) = \begin{cases} \frac{1}{4} + x - \frac{9}{16}x^2 & 0 < x < 2 \\ 0 & \text{otherwise,} \end{cases}$$

$$f_Y(y) = \begin{cases} 3y^2 & 0 < y < 1 \\ 0 & \text{otherwise.} \end{cases}$$

   (b) Compute the covariance $\mathrm{Cov}(X,Y)$, and compute the correlation coefficient $\rho(X,Y)$.

4. Let X and Y have the joint density function

$$f(x,y) = \begin{cases} \frac{1}{2}e^{-(x+\frac{1}{2}y)} & 0 \le x < \infty, \quad 0 \le y < \infty \\ 0 & \text{otherwise.} \end{cases}$$

   Compute the marginal density of X and the marginal density of Y(don't forget the bounds). Are X and Y independent? Compute $P(X < Y)$.

5. Let X and Y have the joint density function

$$f(x,y) = \begin{cases} 6x & 0 < x < y < 1 \\ 0 & \text{otherwise.} \end{cases}$$

   Compute the marginal density of X and the marginal density of Y(don't forget the bounds). Compute $P(X + Y < 1)$.
   Hint: Be careful with the bounds of integration. It may be helpful to draw a picture of the domain of integration.

6. The random variables X and Y have a joint density function given by

$$f(x,y) = \begin{cases} \frac{2}{x}e^{-2x} & 0 \le x \le \infty, \quad 0 \le y \le x, \\ 0 & \text{otherwise.} \end{cases}$$

   Verify that $\mathrm{Cov}(X,Y) = \frac{1}{8}$.
   Hint: $\int_0^\infty x^n e^{-\lambda x} dx = \frac{n!}{\lambda^\alpha}$.

7. Let X be a random variable such that $E[X] = 1$ and $\mathrm{Var}(X) = 5$. Find
   (a) $E[(2 + X)^2]$,
   (b) $\mathrm{Var}(4 + 3X)$.

8. Let X and Y be random variables. Suppose that $\mathrm{Var}(X) = 100, \mathrm{Var}(Y) = 200, \mathrm{Var}(X + Y) = 400$. Find
   (a) $\mathrm{Var}(2X + 3Y)$,
   (b) $\mathrm{Var}(X - Y)$.

9. Compute the moment generating function of the following distributions:
   (a) The uniform distribution from a to b, where a and b are real numbers, and $a < b$.
   (b) The normal distribution with mean 0 and variance 1.
   Hint: Complete the square in the exponent.
   (c) The exponential distribution with mean $1/\lambda$.

10. Let X be a discrete random variable taking on values $1, 2, 3, 4, 5, 6$, and $P(X = i) = 1/6$ for $i = 1, 2, 3, 4, 5, 6$. That is, X is represents the result of a fair die roll. Compute the moment generating function of X. Use the moment generating function to compute $E[X]$ and $\mathrm{Var}(X)$.

11. Find the first two moments and the variance of a random variable X whose moment generating function is given by
$$M_X(t) = \frac{1}{(1-2t)^4}.$$

12. A company insures homes in three cities, J, K and L. Since sufficient distance separates the cities, it is reasonable to assume that the losses occurring in these cities are independent. The moment generating functions for the loss distributions of the cities are
$$M_J(t) = \frac{1}{(1-2t)^3},$$
$$M_K(t) = \frac{1}{(1-2t)^{2.5}},$$
$$M_L(t) = \frac{1}{(1-2t)^{4.5}}.$$
Let $X = J + K + L$ represent the combined losses from the three cities. Calculate $E[X^3]$.

13. Let X and Y be independent random variables, where X is exponential with mean 2, and $Y \sim \Gamma(3, 2)$. That is, the probability density functions of X and Y respectively are
$$f_X(x) = \begin{cases} \frac{1}{2}e^{-\frac{1}{2}x} & 0 < x < \infty \\ 0 & \text{otherwise,} \end{cases}$$
$$f_Y(y) = \begin{cases} \frac{1}{8}y^2 e^{-\frac{1}{2}y} & 0 < y < \infty \\ 0 & \text{otherwise,} \end{cases}$$
Let $S = X + Y$.
   (a) What is the distribution of S?
   (b) Find $E[S]$ and $\mathrm{Var}(S)$.

The following problems require the use of the Central Limit Theorem.

14. A carpenter wants to measure the length of a table. Each measurement is independent of the others, and have a common mean d and common standard-deviation of 0.01 cm. He estimates the length of the table by taking the average of all the measurements. How many measurements does he have to make in order to make 95% sure that his estimate is accurate to within ± 0.001cm if he trusts the Central Limit Theorem?

15. If 30 fair dice are rolled, find the approximate probability that the average number of dots is between 2 and 4.

16. Let $X_1, X_2, \ldots, X_{25}$ be independent random variables, each of which is uniformly distributed on [0, 2], and let $X = \sum_{i=1}^{25} X_i$.

    (a) Find the mean and variance of X.

    (b) Use the Central Limit Theorem to estimate $P(|X - 24| \leq 3)$.

17. Let $X_1, X_2, \ldots, X_{20}$ be independent and identically distributed continuous random variables with probability density function

$$f(x) = \begin{cases} 3x^2 & 0 < x < 1 \\ 0 & \text{otherwise.} \end{cases}$$

    Calculate an approximation to

$$P\left(\sum_{i=1}^{20} X_i < 16\right).$$

18. Student scores on exams given by a certain instructor have mean 74 and standard deviation 14. This instructor is about to give two exams, one to a class of size 25 and the other to a class of size 64.

    (a) Approximate the probability that the average test score in the class of 25 exceeds 80.

    (b) Approximate the probability that the average test score in the class of 64 exceeds 80.

    (c) Approximate the probability that the average test score in the larger class exceeds that of the other class by more than 2.2 points.

    (d) Approximate the probability that the average test score in the smaller class exceeds that of the other class by more than 2.2 points.