Nested Intervals Theorem

Defn. Suppose that, for each k in $\mathbb{N}$, S_k is a set.

$$\text{We say that the sequence } (S_k)_1^\infty \text{ is nested}$$

if and only if

$$\text{for each } k \text{ in } \mathbb{N},\ S_{k+1} \subseteq S_k.$$

Thm. (Nested Intervals Theorem, NIT)
Suppose that $(S_k)_1^\infty$ is a nested sequence of closed bounded intervals in $\mathbb{R}$.
Then the intersection $\cap_{k=1}^\infty S_k$ is not empty.
Proof: We write out the hypothesis more explicitly as

$$\begin{array}{llcl} (H1) & \text{for each } k \text{ in}\mathbb{N},\ S_k & = & [\ell_k\ ,\ r_k] \text{ and } \ell_k \leq r_k \\ (H2) & \text{for each } k \text{ in}\mathbb{N},\ S_{k+1} & \subseteq & S_k. \end{array}$$

We need to show that there is at least one real number x such that

$$(R) \qquad \text{for each } k \text{ in } \mathbb{N},\ x \in S_k.$$

Hypothesis $(H2)$ tells us that for all indices, $\ell_k \leq \ell_{k+1} \leq r_{k+1} \leq r_k$. A quick inductive argument show that

$$\ell_1 \leq \ell_k \leq \ell_{k+1} \leq r_{k+1} \leq r_k \leq r_1.$$

So that the sequence (ℓ_k) is increasing and bounded above by r_1 and the sequence (r_k) is decreasing and bounded below by ℓ_1. Applying the Monotone Convergence Theorem, we get

$$\begin{array}{lcl} \ell_* & \doteqdot & lub\,(\ell_k) = \lim\,(\ell_k) \in \mathbb{R} \qquad \text{and} \\ r_* & \doteqdot & glb\,(r_k) = \lim\,(r_k) \in \mathbb{R}. \end{array}$$

The next step is to show that $\ell_* \leq r_*$. First we consider an arbitrary k and show that ℓ_k is a lower bound for the sequence (r_k). Holding that k øxed, we consider an arbitrary n in $\mathbb{N}$.

If $n = k$, then by $(H1)$ and $(H2)$, $\quad \ell_k \leq r_k = r_n$.
If $n < k$, then by $(H1)$ and $(H2)$, $\quad \ell_k \leq r_k \leq r_n$.
If $n > k$, then by $(H1)$ and $(H2)$, $\quad \ell_k \leq \ell_n \leq r_n$.

Thus for all n, $\ell_k \leq r_n$. It follows that ℓ_k is a lower bound for $(r_n)_{n=1}^\infty$ and $\ell_* \leq glb\,(r_n) = r_*$.
But we made this argument for an arbitrary k. Thus r_* is an upper bound for the set of all ℓ_k and thus $r_* = glb\,(\ell_k) \geq \ell_*$.

What we have proved so far is that for all k, $\ell_k \leq \ell_* \leq r_* \leq r_k$ and thus that

$$\text{for every } k,\ \ell_* \in [\ell_k\ ,\ r_k] \quad \text{and } \ell_* \in \cap_{k=1}^\infty S_k.$$

Remark 1. We have showed even more, namely that

$$[\ell_* \ , \ r_k] \in \cap_{k=1}^{\infty} S_k.$$

If $\ell_* < r_k$ then $\cap_{k=1}^{\infty} S_k$ contains the uncountably many real numbers in the interval $[\ell_*, r_*]$. However, if $\ell_* = r_*$, then the intersection $\cap_{k=1}^{\infty} S_k$ contains the unique element ℓ_*.

Remark 2. Since the intervals are nested, it is easy to see that

$$r_{k+1} - \ell_{k+1} = length(S_{k+1}) \leq length\,(S_k) = r_k - \ell_k$$

for each index. If this sequence of lengths converges to zero, then

$$0 \leq r_* - \ell_* = \lim\,(r_k - \ell_k) = 0$$

and there is a unique real number in the intersection $\cap_{k=1}^{\infty} S_k$.