Homework # 4 , Math 311:02, Fall 2008
Sample Solutions

TASK **§1.3 #25** See Workshop #4

TASK **§1.3 #27** Suppose that $(a_n)_1^\infty$ and $(b_n)_1^\infty$are sequences in $\mathbb{R}$ such that

$$\lim(a_n) = A \neq 0 \quad \text{and} \ \lim(a_n b_n) \ \text{exists.}$$

Show that $(b_n)_1^\infty$ converges.

PROOF: First let's introduce some notation. For each n set $c_n = a_n b_n$. Set $C = \lim(c_n)$.

Since $A \neq 0$, we know that $|A|/2 > 0$. Since $\lim(a_n) = A$ and $|A|/2 > 0$ we get an positive integer N_1 such that

$$(*) \qquad \text{whenever } n \geq N_1 \text{ then also } |a_n - A| < |A|/2.$$

Keep $n \geq N_1$. Then we have

$$|a_n| = |(a_n - A) + A| \geq |A| - |a_n - A| > |A| - |A|/2 = |A|/2 > 0$$

and thus $a_n \neq 0$.

Still keeping $n \geq N_1$,

$$b_n = \frac{c_n}{a_n} =$$

and we expect to prove that

$$\lim(b_n) = \frac{\lim(c_n)}{\lim(a_n)} = \frac{C}{A}$$

Note that we can't appeal to the theorem on limits of products since the entries $1/a_n$ might fail to exist for some small values of the index n. So we keep working with $n \geq N_1$.

$$\begin{aligned}
\left|\frac{c_n}{a_n} - \frac{c}{A}\right| &= \left|\frac{c_n}{a_n} - \frac{C}{a_n} + \frac{C}{a_n} - \frac{C}{A}\right| \\
&\leq \left|\frac{c_n}{a_n} - \frac{C}{a_n}\right| + \left|\frac{C}{a_n} - \frac{C}{A}\right| \\
&\leq \left|\frac{1}{a_n}\right| |c_n - C| + |C| \left|\frac{1}{a_n} - \frac{1}{A}\right| \\
&\leq \frac{2}{|A|} |c_n - C| + |C| \frac{1}{|a_n|} \frac{1}{|A|} |a_n - A| \\
&\leq \frac{2}{|A|} |c_n - C| + |C| \frac{2}{|A|} \frac{1}{|A|} |a_n - A|
\end{aligned}$$

Each of the terms on the right converges to 0.

Finally we start the "$\varepsilon - N$ endgame". Consider an arbitrary postive ε. Note $\varepsilon/2 > 0$. We get positive integers N_2 and N_3 such that

$$\begin{aligned}
\text{whenever } n &\geq N_2 \text{ then } \frac{2}{|A|} |c_n - C| = \left|\frac{2}{|A|} |c_n - C| - 0\right| < \frac{\varepsilon}{2} \text{ and} \\
\text{whenever } n &\geq N_3 \text{ then } |C| \frac{2}{|A|} \frac{1}{|A|} |a_n - A| = \left||C| \frac{2}{|A|} \frac{1}{|A|} |a_n - A| - 0\right| < \frac{\varepsilon}{2}
\end{aligned}$$

Thus whenever $n \geq \max\{N_1, \ N_2, \ N_3\}$ it follows that

$$\left|b_n - \frac{C}{A}\right| = \left|\frac{c_n}{a_n} - \frac{c}{A}\right| \leq \frac{2}{|A|} |c_n - C| + |C| \frac{2}{|A|} \frac{1}{|A|} |a_n - A| < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon$$

TASK **§1.3 #32a,c,e**. Find the limits. Note no proof is required, but a reasonable computation should be provided and annotated.

a) FIND $L_a = \lim\left(\frac{n^2+4n}{n^5-5}\right)$ RESULT L_a =0.2

WORK For each positive integer n

$$\begin{aligned}\left(\frac{n^2+4n}{n^5-5}\right) &= \frac{1+4n^{-1}}{1-5n^{-2}} \text{ after dividing top and bottom by } n^2 \\ &\to \frac{\lim(1+4/n)}{\lim(1-5/n^2)} = \frac{1+4\cdot 0}{5-5\cdot 0} = \frac{1}{5} \text{ as } n\to\infty\end{aligned}$$

c) FIND $L_c = \lim \frac{\sin(n^2)}{\sqrt{n}}$ RESULT $L_c = 0$

WORK Rewrite the n^{th} entry $a_n b_n$ where $a_n = \sin(n^2)$ and $b_n = 1/\sqrt{n}$. We know the sequence (a_n) is bounded. We will check that the sequence (b_n) converges to 0.

For every index n in $\mathbb{N}$,we know that $\sqrt{n} \leq n$ and thus that

$$\left|\frac{1}{\sqrt{n}} - 0\right| = \frac{1}{\sqrt{n}} \leq \frac{1}{n}$$

Since $\lim(1/n) = 0$, the squeeze theorem tells us that $\lim b_n = \lim(1/\sqrt{n}) = 0$.

e) FIND L_e =$\lim(a_n b_n)$ where $a_n = \sqrt{4-\frac{1}{n}} - 2$ and $b_n = n$. RESULT $L_e = -1/4$

WORK The sequence (b_n) is not bounded and thus not convergent. We need to anti-simplify a_n so that we can get control over the product $a_n b_n$.

$$a_n = \frac{\sqrt{4-\frac{1}{n}}-2}{1}\cdot\frac{\sqrt{4-\frac{1}{n}}+2}{\sqrt{4-\frac{1}{n}}+2} = \frac{\left(4-\frac{1}{n}\right)-4}{\sqrt{4-\frac{1}{n}}+2} = \frac{-\frac{1}{n}}{\sqrt{4-\frac{1}{n}}+2}$$

Thus

$$a_n b_n = \frac{-\frac{1}{n}}{\sqrt{4-\frac{1}{n}}+2}\cdot n = \frac{-1}{\sqrt{4-\frac{1}{n}}+2}$$

We have

$$\lim\left(\sqrt{4-\frac{1}{n}}+2\right) = \lim\sqrt{4-\frac{1}{n}}+2 = \sqrt{\lim\left(4-\frac{1}{n}\right)}+2 = \sqrt{4-0}+2 = 2+2$$

and thus

$$\lim(a_n b_n) = \frac{-1}{\lim\left(\sqrt{4-\frac{1}{n}}+2\right)} = \frac{-1}{2+2}$$

TASK §1.2 #14 Prove that every Cauchy sequence is bounded.

PROOF Suppose that $(w_n)_1^\infty$ is a Cauchy sequence. We must produce a positive M such that for all indices $|w_n| \le M$.

Since $(w_n)_1^\infty$ is Cauchy, we know that for every positive ε there exists at least one positive integer H with the property that

$$\text{whenever both } m \ge H \text{ and } n \ge H, \text{ then } |w_n - w_m| < \varepsilon.$$

Set $\varepsilon_0 = 1$. Let H_0 denote a positive integer such that

$$\text{whenever both } m \ge H_0 \text{ and } n \ge H_0, \text{ then } |w_n - w_m| < \varepsilon_0 = 1.$$

Consider an arbitrary n with $n \ge H_0$.

$$|w_n| = |(w_n - w_H) + w_H| \le |w_n - w_H| + |w_H| < 1 + |w_H|.$$

Next set

$$A = \max\left(\{|w_n| : n \le H\}\right)$$

and

$$M = \max\left(\{A, 1 + |w_H|\}\right).$$

Since $M \ge 1 + |w_h| \ge 1$, we get $M > 0$. It remains to verify that $|w_n| \le M$ for all indices n. Consider an arbitrary n. If $n \le H$, then $|w_n| \le A \le M$. If $n > H$, then $|w_n| \le 1 + |w_H| \le M$.

So we did find positive M such that $|w_n| \le M$ for all indices.

TASK **§1.2 #16** Prove, directly from the definition, that the product of Cauchy sequences is Cauchy.

PROOF Suppose that $(a_n)$and (b_n) are Cauchy sequences. Set $c_n = a_n b_n$.

Consider an arbitrary postive ε. We seek H so that

$$\text{whenever both } n \ge H \text{ and } m \ge H, \text{ then } |c_n - c_m| < \varepsilon.$$

Consider arbitrary m, n.First we look for an estimate.

$$\begin{aligned} |c_n - c_m| &= |a_n b_n - a_m b_m| = |a_n (b_n - b_m) + a_n b_m - a_m b_m| \\ &\le |a_n (b_n - b_m)| + |a_n - a_m| |b_m| \end{aligned}$$

Since the sequences (a_n) and (b_n) are Cauchy, we know they are bounded by §1.2 #14. Thus we get positive constants M_a and M_b such that

$$\text{for all indices, } |a_n| \le M_a \quad \text{and} \quad |b_m| \le M_b.$$

It follows that

$$\text{for all indices, } |c_n - c_m| \le M_a |b_n - b_m| + M_b |a_n - a_m|$$

Note that $\varepsilon/2M_a$ and $\varepsilon/2M_b$ are positive. Since the sequences (a_n) and (b_n) are Cauchy, we get positive integers H_a and H_b such that

$$\begin{aligned} \text{whenever both } n &\ge H_a \text{ and } m \ge H_a, \text{ then } |a_n - a_m| < \varepsilon/2M_a \\ \text{whenever both } n &\ge H_b \text{ and } m \ge H_b, \text{ then } |b_n - b_m| < \varepsilon/2M_b \end{aligned}$$

Now take $H = \max(H_a, H_b)$. Suppose both $n \ge H$ and $m \ge H$. Then

$$|a_n - a_m| < \varepsilon/2M_a \quad \text{and} \quad |b_n - b_m| < \varepsilon/2M_b$$

so

$$\begin{aligned} |c_n - c_m| &\le M_a |b_n - b_m| + M_b |a_n - a_m| \\ &< M_a \frac{\varepsilon}{2M_a} + M_b \frac{\varepsilon}{2M_b} = \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon. \end{aligned}$$

TASK **§1.2 #22a)** Suppose that S is a non-empty set in $\mathbb{R}$ which is bounded above. Set $a = lub(S)$. Show that if a is not an element of S then a must be an accumulation point of S.
#22b) Suppose that S is a non-empty set in $\mathbb{R}$ which is bounded below. Set $b = glb(S)$. Show that if b is not an element of S then b must be an accumulation point of S.

EXPLORATION Recall that z is an accumulation point of S if and only if for every positive ε the set $S \cap V_\varepsilon(z)$ contains infinitely many elements.

PROOF (a) Assume that $a \notin S$. Consider an arbitrary postive ε. Note that for every n in $\mathbb{N}$, $\varepsilon/n > 0$. Define a sequence $(s_n)_1^\infty$ as follows.

Since $a - \varepsilon/1 < a$, we know that $a - \varepsilon/1$ is not an upper bound for S. We can, and do, pick s_1 in S so that $a - \varepsilon/1 < s_1 \le a$. Since $s_1 \in S$ and $a \notin S$ we note that $s_1 \ne a$ so

$$a - \varepsilon/1 < s_1 < a.$$

Suppose that $k \in \mathbb{N}$ and that we have found s_n for $1 \le n \le k$ so that

$$\begin{aligned} s_1 &< s_2 < ... < s_k \text{ and} \\ \text{whenever } 1 &\le n \le k, \text{ then } s_n \in S \text{ and } a - \varepsilon/n < s_n < a \end{aligned}$$

Thus $\max(s_k, a - \varepsilon/k + 1) < a$ and $\max(s_k, a - \varepsilon/k + 1)$ cannot be an upper bound for S. Pick s_{k+1} in S so that

$$\max(s_k, a - \varepsilon/(k+1)) < s_{k+1} \le a$$

Note that $s_{k+1} \ne a$ since $s_{k+1} \in S$ and $a \notin S$. Thus we have $s_k < s_{k+1}$ and $a - \frac{\varepsilon}{k+1} < s_{k+1} < a$.

Since the sequence $(s_k)_1^\infty$ is strictly increasing, there are infinitely many elements in the set $\{s_k : k \in \mathbb{N}\}$. For all k, $\varepsilon/k \le \varepsilon$ and so $a - \varepsilon \le a - \varepsilon/k < s_k < a$ and thus $s_k \in V_\varepsilon(a)$. For all indices, $s_k \in S$. Thus $\{s_k : k \in \mathbb{N}\}$ is an infinite subset of $S \cap V_\varepsilon(a)$. Equivalently, there are infinitely many elements of S in $V_\varepsilon(a)$. Since ε was arbitrary, a is an accumulation point of S.

(b) Left as an exercise.

TASK **§1.2 #24** Consider a sequence $(a_n)_1^\infty$ which has limit A. Consider the set of entries in that sequence,namely $S = \{a_n : n \in \mathbb{N}\}$. Assume that S is an infinite set. Show that A is an accumulation point of S.

PROOF Consider an arbitrary positive ε. It is necessary to show that there are infinitely many elements of S in $V_\varepsilon(A)$. To do this I will display a one-to-one function f from $\mathbb{N}$ into $S \cap V_\varepsilon(A)$. The image of this function will be an infinite set of elements of S that are also in $V_\varepsilon(A)$. The function f is defined inductively.

Since $\varepsilon > 0$ we get N_1 such that whenever $n \ge N_1$ then $|a_n - A| < \varepsilon$. Since $\{a_n : n < N_1\}$ is a finite set, the set $\{a_n : n \ge N_1\}$ must be an infinite set. So there must be an element in this set different from A. Pick n_1 so that $n_1 \ge N_1$ and $a_{n_1} \ne A$. Set $s_1 = a_{n_1}$.

Set $\varepsilon_2 = |A - s_1|$. Note that $0 < \varepsilon_2 < \varepsilon$ since $s_1 \ne A$. We get N_2 such that whenever $n \ge N_2$ then $|a_n - A| < \varepsilon_2$. The set $\{a_n : n \ge N_2\}$ must be an infinite set. So there must be an element in this set not in $\{A, s_1\}$. Pick n_2 so that $n_2 \ge N_2$ and $a_{n_2} \notin \{A, s_1\}$. Set $s_2 = a_{n_2}$. Note that we have $\varepsilon > |A - s_1| = \varepsilon_2 > |A - s_2| > 0$

Suppose $K \in \mathbb{N}$ and we have already picked positive integers n_k for all $k \le K$ and set $s_k = a_{n_k}$ so that

The finite sequence $(|A - s_k|)_{k=1}^K$ is strictly decreasing in the positive reals.

Set $\varepsilon_{K+1} = |A - s_K|$. Since $\varepsilon_{K+1} > 0$ we can pick N_{K+1} so that whenever $n \geq N_{K+1}$ then $|A - a_n| < \varepsilon_{K+1}$. Since the set $\{a_n : n \geq N_{K+1}\}$ must be an infinite set it must contain an element not in $\{A, s_1, s_2, ..., s_K\}$. Pick n_{K+1} so that $n_{K+1} \geq N_{K+1}$ and $a_{n_{K+1}} \notin \{A, s_1, s_2, ..., s_K\}$. Set $s_{K+1} = a_{n_{K+1}}$.

Thus we have defined inductively the sequence $(s_k)_{k \in \mathbb{N}}$. Furthermore we know that the distances $(|A - s_k|)_{k=1}^{\infty}$ are strictly decreasing and positive. Thus if $i < j$ we know that $s_i \neq s_j$ since $|A - s_j| < |A - s_i|$. Thus the function f defined by $f(k) = s_k$ maps $\mathbb{N}$ one-one into S. But we know more. When $n \geq 1$ we have $|A - s_n| \leq |A - s_1| < \varepsilon$. So the image of the one-one function f is in $S \cap V_\varepsilon(A)$.

We have shown that for each positive ε there are infinitely many elements of S in $V_\varepsilon(A)$ and thus that A is an accumulation point of S.

TASK **§1.4 #37** Show that if a sequence is decreasing and bounded below then it converges.
PROOF This is analogous to the proof of the MCT done in class.

TASK **§1.4 #38** Suppose that $c > 1$. Show that $(\sqrt[n]{c})_{n=1}^{\infty}$ converges to 1.

PROOF.

Consider an arbitrary index n. We have $\sqrt[n]{c} > 1$ since

$$\sqrt[n]{c} > 1 \iff c > 1^n, \text{ which is true in this case.}$$

Note that for all n

$$\sqrt[n+1]{c} < \sqrt[n]{c} \iff c < \left(\sqrt[n]{c}\right)^{n+1} \iff c < c^{\frac{n+1}{n}} = c^{1+\frac{1}{n}} = c \cdot \sqrt[n]{c}$$

Since we have seen that $\sqrt[n]{c} > 1$ in this case, we conclude that $c < c \cdot \sqrt[n]{c}$ and thus that $\sqrt[n+1]{c} < \sqrt[n]{c}$.

Since our sequence $(\sqrt[n]{c})$ is decreasing and bounded below by 1, we conclude that our sequence converges and

$$L = \lim \sqrt[n]{c} = glb\left(\sqrt[n]{c}\right) \geq 1.$$

We can conclude that $L = 1$ by showing that $L > 1$ implies something false.

Suppose that $L > 1$. Set $p = L - 1$ so that $p > 0$. Now for all n

$$\sqrt[n]{c} \geq L = 1 + p \quad \text{and thus} \quad c \geq (1 + p)^n$$

But for all n

$$(1 + p)^n \geq 1 + np.$$

So for all n

$$c \geq 1 + np \ \text{ and } \ \frac{c - 1}{p} \geq n.$$

Thus $(c - 1)/p$ is an upper bound for $\mathbb{N}$, which must be false. □

TASK **§1.4 #39** Suppose that $\lim(x_n) = L = \lim(y_n)$. Define $(z_n)_1^{\infty}$ by

$$z_{2n} = x_n \quad \text{and} \quad z_{2n-1} = y_n \quad \text{for each positive integer } n$$

or equivalently

$$z_k = \begin{cases} x_{k/2} & \text{for ever } k \text{ in } \mathbb{N} \\ y_{(k+1)/2} & \text{for odd } k \text{ in } \mathbb{N} \end{cases}$$

Show that $\lim(z_n) = L$.

PROOF. Consider a positive ε. We need to find a positive integer K such that

$$(*) \qquad \text{whenever } k \geq K \text{ then } |z_k - L| < \varepsilon$$

By hypothesis we can get positive integers N_x and N_y such that

$$\begin{aligned} \text{whenever } n &\geq N_x \text{ then } |x_n - L| < \varepsilon \\ \text{whenever } n &\geq N_y \text{ then } |y_n - L| < \varepsilon \end{aligned}$$

Note that for all even k in $\mathbb{N}$

$$\begin{aligned} |z_k - L| &< \varepsilon \iff \left|x_{k/2} - L\right| < \varepsilon \\ \text{and } \left|x_{k/2} - L\right| &< \varepsilon \text{ whenever } k/2 \geq N_x \text{ and thus whenever } k \geq 2N_x \end{aligned}$$

Also note that for all odd k in $\mathbb{N}$

$$\begin{aligned} |z_k - L| &< \varepsilon \iff \left|y_{(k+1)/2} - L\right| < \varepsilon \\ \text{and } \left|y_{(k+1)/2} - L\right| &< \varepsilon \text{ whenever } (k+1)/2 \geq N_y \text{ and thus whenever } k \geq 2N_y - 1. \end{aligned}$$

To satisfy condition $(*)$ it is sufficient to take

$$k \geq N = \max\left(2N_x\ ,\ 2N_y - 1\right) \qquad \square$$

TASK **§1.4 #40** Define a sequence $(a_n)_1^\infty$ inductively by setting

$$\begin{aligned} a_n &= 6 \text{ if } n = 1 \quad \text{and} \\ a_n &= \sqrt{6 + a_{n-1}} \text{ if } n > 1. \end{aligned}$$

Show that this sequence is convergent and find its limit.

RESULT $\lim (a_n) = 3$

WORK To get an idea of what is going on compute the first few entries

$$\begin{aligned} a_1 &= 6 \\ a_2 &= \sqrt{6 + a_1} = \sqrt{12} = 2\sqrt{3} \approx 2 \times 1.732 = 3.464 \\ a_3 &= \sqrt{6 + a_2} = \sqrt{6 + 2\sqrt{3}} \approx \sqrt{6 + 3.464} = \sqrt{9.464} \approx 3.078 \end{aligned}$$

We will conjecture that this sequence is strictly decreasing and converges to 3.

PROOF.

Step 1 Verify the monotonicity (by induction). We have already seen that $a_2 < a_1$. Suppose that $n \in \mathbb{N}$ and $a_{n+1} < a_n$. Now

$$a_{n+2} < a_{n+1} \iff \sqrt{6 + a_{n+1}} < \sqrt{6 + a_n} \iff 6 + a_{n+1} < 6 + a_n \iff a_{n+1} < a_n$$

This last inequality is true by the induction hypothesis, so the first is also true.

Step 2. Show that for each n, $3 < a_n$. Again we use a proof by induction. We have already seen that $a_1 = 6 > 3$. Suppose that $n \in \mathbb{N}$ and $a_n > 3$. Now

$$a_{n+1} > 3 \iff \sqrt{6 + a_n} > 3 \iff 6 + a_n > 9 \iff a_n > 3$$

The last inequality is true by the induction hypotheses, so the first inequality is also true.

Step 3 Show that $lim(a_n) = 3$. By the monotone convergence theorem and the result of Step 1, we know that $\lim (a_n)$ exists and equals $glb(a_n)$. Step 2 tells us that 3 is one lower bound for our sequence. Thus $glb\,(a_n) \geq 3$. To show that $glb\,(a_n) = 3$, it is sufficient to show that $glb\,(a_n) \not> 3$.

Suppose that $glb\,(a_n) > 3$ – I will derive something false. Set

$$p = glb\,(a_n) - 3$$

and note that $p > 0$. So for all n we get the following conclusions

$$a_{n+1} \geq 3+p \qquad \sqrt{6+a_n} \geq 3+p \qquad 6+a_n \geq 9+2p+p^2 \qquad a_n \geq 3+2p+p^2$$

Thus $3+2p+p^2$ is a lower bound for our sequence (a_n). Thus

$$\begin{aligned} 3+2p+p^2 &\leq glb\,(a_n) = 3+p \\ 2p+p^2 &\leq p \\ p^2 &\leq -p < 0 \end{aligned}$$

but this last inequality must be false since the square of a positive number must be positive.

We need to find a postive integer N such that whenever $n \geq N$ then also $|b_n - A| < \varepsilon$.

By $(H1)$ we get a positive integer N_a such that whenever $n \geq N_a$ then $|a_n - A| < \varepsilon$. By $(H1)$ we also get a positive integer N_c such that whenever $n \geq N_c$ then $|c_n - A| < \varepsilon$. Now by $(H2)$ we learn that if **both** $n \geq N_a$ **and** $n \geq N_c$ we get

$$\begin{aligned} A-\varepsilon &< a_n \leq b_n \leq c_n < A+\varepsilon, \text{ which tells us that} \\ -\varepsilon &< b_n - A < \varepsilon \ \text{ and finally } \ |b_n - A| < \varepsilon. \end{aligned}$$

So what positive integer should we take for N? We want N big enough that whenever $n \geq N$, then also $n \geq N_a$ and $n \geq N_c$. We can take $N = \max(N_a,\ N_c)$.