Homework Sets # 9 , Math 311:02, Fall 2008
Sample Solutions

§4.1 #1 Task Consider the function defined by $f(x) = x^2$ and an arbitrary point (x_0, y_0) on the graph of f subject only to the condition $x_0 \neq 0$. Without using the notion of derivative find the equation of a straight line that intersects this graph only at (x_0, y_0).
Result: $y = (2x_0)(x - x_0) + (x_0)^2$.
Work: Since (x_0, y_0) is on the graph of f we know $y_0 = (x_0)^2$. Thus, for each real m, the line $y = m(x - x_0) + (x_0)^2$ intersects the graph of f at (x_0, y_0). We need to find m so that the line cannot intersect the graph of f anywhere else. So we look for m such that the quadratic equation

$$x^2 - \left(m(x - x_0) + (x_0)^2\right) = 0$$

has only the one (repeated) real solution x_0. Put this quadratic equation into standard form

$$x^2 + [-m]\,x + \left[mx_0 - (x_0)^2\right] = 0$$

The quadratic equation gives solutions

$$\frac{-[-m] \pm \sqrt{[-m]^2 - 4 \cdot 1 \cdot \left[mx_0 - (x_0)^2\right]}}{2 \cdot 1}$$

To get a repeated root x_0 we need

$$x_0 = \frac{m}{2} \qquad \text{and} \qquad m^2 - 4mx_0 + 4\,(x_0)^2 = 0$$

or equivalently

$$m = 2x_0 \qquad \text{and} \qquad (m - 2x_0)^2 = 0$$

which reduces to the condition we knew we needed namely, $m = 2x_0$.

§4.1 #2 Task Suppose that

$$f \text{ is a function; } c \text{ is an accumulationpoint of } dom(f);\ c \in dom(f);\ L \in \mathbb{R}$$

Prove the equivalence of the statements

$$(1)\ \lim_{x \to c} \frac{f(x) - f(c)}{x - c} = L \quad \text{and} \quad (2)\ \lim_{h \to 0} \frac{f(c+h) - f(c)}{h} = L$$

Proof Define the function Q by

$$Q(h) = \frac{f(c+h) - f(c)}{h}$$

and note that

$$Dom(Q) = \{h : c + h \in Dom(f) \text{ and } h \neq 0\}$$

Step 1 Show that (1) implies (2)

Assume (1). Since c is an accumulation point of $Dom(f)$ we get a sequence $(x_n)_{n\in\mathbb{N}}$ in $Dom(f) - \{c\}$ such that.$\lim(x_n) = c$. For each n define $h_n = x_n - c$. Note that for all n, $h_n \in Dom(Q) - \{0\}$ and $\lim(h_n) = 0$. This verifies that 0 is an accumulation point of $Dom(Q)$.

Now we can use the sequential criterion for functional convergence to verify the limit statement (2). Consider an arbitrary sequence $(z_n)_{n\in\mathbb{N}}$ in $Dom(Q) - \{0\}$ and assume that $\lim(z_n) = 0$. It is easy to check that the sequence $(c + z_n)_{n\in\mathbb{N}}$ is a sequence in $Dom(f) - \{c\}$ and that $\lim(c + z_n) = c$. By (1) and some arithmetic in the bottom of the fractions, we know that

$$\lim \frac{f(c+z_n) - f(c)}{z_n} = \lim \frac{f(c+z_n) - f(c)}{(c+z_n) - c} = L$$

This is just what we need to conclude that

$$(2)\quad \lim \frac{f(c+h) - f(c)}{h} = L$$

Step 2 Show that (2) implies (1)

Assume (2). Note that in this direction our hypothesis takes explicit care of the domain and accumulation point issues. We use an $\varepsilon\delta$-proof to finish (1). Consider an arbitrary positive ε. Use (2) to pick a positive δ with the property that

$$\text{for all } h \text{ in } Dom(Q),\ \ 0 < |h - 0| < \delta \Rightarrow |Q(h) - L| < \varepsilon.$$

Now consider an arbitrary x in $Dom(f)$ and assume that $0 < |x - c| < \delta$. We immediately get

$$\left| \frac{f(x) - f(c)}{x - c} - L \right| = |Q(x - c) - L| < \varepsilon.$$

So (1) is verified.

§4.1 #4 Task Let $g(x) = x^2$. Consider arbitrary real c. Use the definition of derivative to compute $g'(c)$.
Proof There is no problem with verifying that c is in the domain of g and that c is an accumulation point of the domain of g. Now compute for $x \neq c$.

$$\frac{g(x) - g(c)}{x - c} = \frac{x^2 - c^2}{x - c} = \frac{x - c}{x - c}\frac{x + c}{1} = x + c \to 2c \quad \text{as } x \to c.$$

§4.1 #5 Task Set

$$h(x) = \begin{cases} x^3 \ \sin\left(\frac{1}{x}\right) & \text{if} \quad x \neq 0 \\ 0 & \text{if} \quad x = 0 \end{cases}$$

Use the computational rules of derivatives from Calc I and II.
a) Find the derivative of h
b) Show that h' is continuous on $\mathbb{R}$
c) Show that there is one real c such that h' is differentiable on $\mathbb{R} - \{c\}$ but not at .

Result (a)

$$h'(x) = \begin{cases} 3x^2 \sin\left(\frac{1}{x}\right) - x\cos(\frac{1}{x}) & \text{if} \quad x \neq 0 \\ 0 & \text{if} \quad x = 0 \end{cases}$$

Work for (a) Using the algebra of derivatives we get $h'(= c)$ easily for non-zero c

$$h'(c) = 3c^2 \sin\left(\frac{1}{c}\right) - c^3 \cos(\frac{1}{c})c^{-2}$$

For $c = 0$ it is convenient to use the alternate defintion

$$h'(0) = \lim_{t \to 0} \frac{h(0 + t) - h(0)}{t} = \lim_{t \to 0} \ t^2 \sin\left(\frac{1}{t}\right) = 0$$

since the factor t^2 has limit 0 and the second factor is bounded.
Proof for (b) Freshman calculus gives us a derivative for h' at every non-zero c. So h' must be continuous at every non-zero c. To see that h' is continuous at 0 note that

$$h'(x) - h'(0) = h'(x) = x\,[3x\sin(1/x)] - \cos(1/x)$$

$$|h'(x) - h'(0)| = |x| \cdot |3x\sin(1/x) - \cos(1/x)| \leq |x - 0| \cdot (3\,|x| + 1)$$

Thus

$$0 \leq \lim_{x \to 0} \ |h'(x) - h'(0)| \leq \lim_{x \to 0} \ |x - 0| \cdot (3\,|x| + 1) = 0$$

and we get the continuity at zero by the Squeeze Theorem.

Result and Proof for (c) As noted above h' will be differentiable at every non-zero c, where
$h'(c) = 3c^2 \sin\left(\frac{1}{c}\right) - c^3 \cos(\frac{1}{c})c^{-2} = 3c^2 \sin\left(\frac{1}{c}\right) - c\cos(\frac{1}{c})$

$$\begin{aligned} h''(c) &= \left[6c\sin(1/c) - 3c^2\cos(1/c)c^{-2}\right] - \left[\cos(1/c) - c\sin(1/c)c^{-2}\right] \\ &= \left[6c + c^{-1}\right]\sin(1/c) - [3 - 1]\cos(1/c) \end{aligned}$$

We must try to take a limit of the difference quotient at zero explicitly

$$\begin{aligned} \frac{h'(x) - h'(0)}{x - 0} &= \frac{(x\,[3x\sin(1/x)] - \cos(1/x)) - 0}{x} \\ &= \left(3x\sin(1/x) - \frac{\cos(1/x)}{x}\right) \end{aligned}$$

The first term converges to 0 as x runs to 0. But the second term is not bounded in any neighborhood of 0 so the second term cannot converge. If $h''(0)$ exists then the second term must converge. So $h''(0)$ cannot exist.

§4.1 #7 Task Suppose that f is defined on (a, b). We say that

"f satisfies a Lipschitz condition at c in (a, b)"

iff

there is a positive M and a positive r such that

for all x in (a, b) $\quad |x - c| < r \Rightarrow |f(x) - f(c)| < M|x - c|$

NOTE that I have translated this definition so that it agrees with the definition I gave earlier for "local Lipschitz condition".
a) Give an example of a function that is continous at c but fails to satisfy a Lipschitz condition at c
b) Show that if a function is differentiable at c then it does satisfy a Lipschitz condition.

a) In light of (b) we look for a continuous function that fails to have a derivative at c. The absolute value function is tempting, but does satisfy a Lipschitz condition. So try $f(x) = x^{1/3}$ on the interval $(a, b) = (-1, 1)$. Since f is the inverse of a strictly increasing function, it must be continuous everywhere.

Suppose, for the sake of contradiction, that f does satisfy a Lipschitz condition at 0 with M and r as in the defintion. Then for all x near 0 but not equal to 0

$$M > \left| \frac{f(x) - f(0)}{x - 0} \right| = \left| \frac{x^{1/3} - 0}{x - 0} \right| = \left| \frac{1}{x^{2/3}} \right|$$

Now look at the sequence $(x_n)_{n \in \mathbb{N}}$ defined by

$$x_n = \frac{r}{2} \frac{1}{n^{3/2}} = \frac{r}{2} n^{-3/2}$$

Since we have for all indices

$$-1 < 0 < x_n < \frac{r}{2} \frac{1}{n^1} < \min(r, 1).$$

So for all indices

$$M > \left| \frac{1}{(x_n)^{2/3}} \right| = \left| \frac{1}{\left(\frac{r}{2} \cdot n^{-3/2}\right)^{2/3}} \right| = \left(\frac{r}{2}\right)^{-2/3} \cdot n$$

which says that $(r/2)^{2/3} M$ is an upper bound for $\mathbb{N}$, which is false.

§4.1 #11 Task Define f on $(0,1)$ by

$$f(x) = \sqrt{2x^2 - 3x + 6}$$

show that f is differentiable at every point of its domain. Find the derivative.

Proof Let r denote the square root function, $r(u) = \sqrt{u}$. We know that r is defined on $[0, +\infty)$ and is differentiable on $(0, +\infty)$ with $r'(x) = x^{-1/2}$ for positive x.

Let P denote the polynomial $P(x) = 2x^2 - 3x + 6$. We know that P is continuous and differentiable everywhere.

Note that $f = r \circ P$. We check to see if we can apply the Chain Rule.

Suppose c is in $(0,1)$. Then c is both in the domain of f and is an accumulation point for the domain of f. Our "inside function" P is differentiable at c. Our "outside function" r is differentiable at c provided that $P(c) > 0$.Note that

$$\begin{aligned} P(c) &= 2c^2 - 3c + 6 \\ &> -3c + 6 \text{ since } c > 0 \text{ and thus } 2c^2 > 0 \\ &> 6 - 3 = 3 > 0 \text{ since } c < 1 \text{ and thus } -3c > -3 \end{aligned}$$

So the Chain Rule Theorem tells us that $f\,'$ exists and that

$$f\,'(c) = r'(P(c))P'(c) = \frac{1}{2}\left(2c^2 - 3c + 6\right)(4c - 3)$$

§4.1 #13 Task Suppose that function f maps $[a,b]$ into $[c,d]$ and function g maps $[c,d]$ into $\mathbb{R}$. Suppose further that f and $f\,'$ are differentiable on $[a,b]$ and that g and g' are differentiable on $[c.d]$. Show that $g \circ f$ is differentiable, that $(g \circ f)'$ is also differentiable and compute the derivatives.

Results and Proof The hypotheses are sufficient to satisfy the hypotheses of the Chain Rule Theorem applied to $g \circ f$. Thus on $[a,b]$

$$(g \circ f)'(x) = g'(f(x)) \cdot f\,'(x).$$

Next we need to see why $(g \circ f)'$ has a derivative. First note that $(g \circ f)'$ is a product of two functions. To apply the product rule we need to check that each factor is differentiable.

By hypotheses the second factor $f\,'$ is differentiable. The first factor is the composition of g' following f. We are told that f maps $[a,b]$ differentiably into $[c,d]$ and that g' maps $[c,d]$ differentiably into $\mathbb{R}$. So by the Chain Rule Theorem, $(g' \circ f)$ is differentiable and

$$(g' \circ f)'(x) = g''(f(x)) \cdot f\,'(x)$$

Now we know that we can apply the product rule to get

$$\begin{aligned} (g \circ f)''(x) &= \left[(g' \circ f)'(x)\right] \cdot f\,'(x) + (g' \circ f)(x) \cdot [f\,''(x)] \\ & [g\,''(f(x)) \cdot f\,'(x)] \cdot f\,'(x) + g'(f(x)) \cdot [f\,''(x)] \end{aligned}$$

§4.1 #14 Task Suppose that f is differentiable on $\mathbb{R}$ and that g is defined by

$$g(x) = x^2 f(x^3)$$

Show that g is differentiable and find $g'(x)$.
Result $g'(x) = 2xf(x^3) + 3x^4 f\,'(x^3)$.
Proof. We use our usual notation for power functions

$$P_n(x) = x^n.$$

Then we see that g is the product of the function P_2 and the composition $f \circ P_3$. Since f and P_3 are both differentiable on $\mathbb{R}$ we get

$$(f \circ P_3)'(x) = f\ '(P_3(x)) \cdot P_3'(x) = f\ '\left(x^3\right) \cdot 3x^2$$

We also know that

$$P_2'(x) = 2x$$

Thus

$$\begin{aligned} g'(x) &= P_2'(x) \cdot (f \circ P_3)(x) + P_2(x) \cdot (f \circ P_3)'(x) \\ &= 2x \cdot f(x^3) + x^2 \cdot \left[f\ '\left(x^3\right) \cdot 3x^2\right] \\ &= 2x \cdot f(x^3) + 3x^4 \cdot f\ '\left(x^3\right) \end{aligned}$$