Homework Sets # 7 and #8 , Math 311:02, Fall 2008

Sample Solutions

§3.1 #2 Task Define function f on $[-4, 0]$ by

$$f(x) = \begin{cases} \frac{2x^2-18}{x+3} & \text{if} \quad x \neq -3 \\ -12 & \text{if} \quad x = -3 \end{cases}$$

Show that f is continuous at -3.

Proof Note that -3 is an accumulation point of $Dom(f)$ and $-3 \in Dom(f)$. To prove that f is continuous at -3 in this situation it is sufficient to prove that $\lim_{x\to-3} f(x) = f(-3)$.

When $x \in Dom(f) - \{-3\}$ we have

$$f(x) = \frac{2x^2 - 18}{x+3} = \frac{2(x-3)(x+3)}{x+3} = 2(x-3)\frac{x+3}{x+3} = 2x - 6$$

since $x + 3 \neq 0$. The polynomial $2x - 6$ is continuous everywhere, and in particular at -3. So

$$\lim_{x\to-3} f(x) = \lim_{x\to-3} (2x-6) = 2(-3) - 6 = -12$$

Since $f(-3) = -12$, we have shown

$$\lim_{x\to-3} f(x) = -12 = f(-3)$$

and it follows that f is continuous at -3.

§3.1 #4 Done in class. Don't grade.

§3.1 #6 Task Let $f(x) = \sqrt{x}$. Show that f is continuous at each p in $Dom(f)$.

Proof We know that $Dom(f)$ is the set of non-negative reals. Suppose that $p \geq 0$. Consider an arbitrary positive ε. We need to provide a positive δ with the property that

$$\text{for all non-negative } x,\ |x-p| < \delta \Rightarrow \left|\sqrt{x} - \sqrt{p}\right| < \varepsilon. \qquad (6.1)$$

We treat the case where $p = 0$ separately from the case $p > 0$.

Case 1: $p = 0$

Since $p = 0$, we know that, for all non-negative x,

$$\left|\sqrt{x} - \sqrt{p}\right| < \varepsilon \Leftrightarrow \sqrt{x} < \varepsilon \Leftrightarrow 0 \leq \sqrt{x} < \varepsilon \Leftrightarrow 0 \leq x < \varepsilon^2.$$

So we take $\delta = \varepsilon^2$. Clearly δ is positive and condition (6.1) is satisfied.

Case 2: $p > 0$

For non-negative x we have

$$\sqrt{x} - \sqrt{p} = \frac{\sqrt{x} - \sqrt{p}}{1} \cdot \frac{\sqrt{x} + \sqrt{p}}{\sqrt{x} + \sqrt{p}} = \frac{x-p}{\sqrt{x} + \sqrt{p}}.$$

From this we get a global Lipschitz condition at p

$$\left|\sqrt{x} - \sqrt{p}\right| = \left|\frac{x-p}{\sqrt{x} + \sqrt{p}}\right| \leq \frac{|x-p|}{\sqrt{p}} = \frac{1}{\sqrt{p}} \cdot |x-p|.$$

Note that

$$\frac{1}{\sqrt{p}} \cdot |x-p| < \varepsilon \Leftrightarrow |x-p| < \sqrt{p} \cdot \varepsilon.$$

Take $\delta = \sqrt{p} \cdot \varepsilon$. Now consider arbitrary x in $Dom(f)$. Assume that $|x - p| < \delta$. Then

$$\left|\sqrt{x} - \sqrt{p}\right| \leq \frac{1}{\sqrt{p}} \cdot |x - p| < \frac{1}{\sqrt{p}}.\delta = \varepsilon.$$

§3.1 #9 TASK Define f on $(0, 1)$ by $f(x) = x \sin(1/x)$. Can one define $f(0)$ so that f is continuous at 0? Explain.
Result Yes, if one defines $f(0) = 0$ then the resulting function f with Dom(f) now equal to the interval $[0, 1)$ will be continuous at 0.
Proof For x in the enlarged domain $[0, 1)$ we get the estimate

$$x \neq 0 \Rightarrow |f(x) - f(0)| = |x \sin(1/x) - 0| = |x| \cdot |\sin(1/x)| \leq |x| = |x - 0|$$

and thus we get a Lipschitz condition at 0 :

$$|f(x) - f(0)| \leq 1 \cdot |x - 0|$$

Continuity at 0 follows.

§3.2 #14 TASK Suppose that f is continuous at p in $Dom(f)$. Set $g(x) = |f(x)|$. Show that g is continuous at p.

Proof. Consider an arbitrary postive ε. Since f is continuous at p, we can and do pick a postive δ such that, for all x in $Dom(f)$

$$|x - p| < \delta \Rightarrow |f(x) - f(p)| < \varepsilon. \tag{14.1}$$

Use this δ to work with g. Note that $Dom(g) = Dom(f)$.
Consider arbitrary x in $Dom(g)$. So we know $x \in Dom(f)$.
Assume that $|x - p| < \delta$.
Recall that for all real a and b

$$|\ |a| - |b|\ | \leq |a - b|\,.$$

Thus

$$|g(x) - g(p)| = |\ |f(x)| - |f(p)|\ | \leq |f(x) - f(p)| < \varepsilon$$

by using (14.1).