Homework # 6 , Math 311:02, Fall 2008
Sample Solutions

§2.1 #16 Task Define f by $Dom(f) = (0,1)$ and

$$f(x) = \frac{x^3 + 6x^2 + x}{x^2 - 6x}$$

Prove that f has a limit at 0 and find that limit.

Result $\lim_{x \to 0} f(x) = -1/6$.

Proof For x in the domain, we know $x \neq 0$ and $x \neq 6$, so that

$$f(x) = \frac{x\left(x^2 + 6x + 1\right)}{x(x-6)} = \frac{x\left(x^2 + 6x + 1\right)}{x\,(x-6)} = \frac{x}{x}\frac{x^2 + 6x + 1}{x-6} = \frac{x^2 + 6x + 1}{x-6}$$

Apply the theorem on limits of polynomials

$$\begin{aligned} \lim_{x \to 0} \left(x^2 + 6x + 1\right) &= 0 + 0 + 1 = 1 \\ \lim_{x \to 0} (x - 6) &= 0 - 6 = -6 \end{aligned}$$

Note that $-6 \neq 0$. Apply the theorem on limits of quotients

$$\lim_{x \to 0} f(x) = \lim_{x \to 0} \frac{x^2 + 6x + 1}{x - 6} = \frac{\lim_{x \to 0}(x^2 + 6x + 1)}{\lim_{x \to 0}(x - 6)} = \frac{1}{-6} = -\frac{1}{6}.$$

§2.1 #19 Task Define f by $Dom(f) = (0,1)$ and

$$f(x) = \frac{\sqrt{9 - x} - 3}{x}$$

Prove that f has a limit at 0 and find that limit.

Result $\lim_{x \to 0} f(x) = -1/6$.
Proof We cannot immediately apply the theorem on limits of quotients since the limit of the bottom is zero. We try to re-express this function so that we can see a factor of x on top. This is the "rationalize the numerator" game.

$$f(x) = \frac{\sqrt{9 - x} - 3}{x}\frac{\sqrt{9 - x} + 3}{\sqrt{9 - x} + 3} = \frac{(9 - x) - 9}{x\left(\sqrt{9 - x} + 3\right)} = \frac{x}{x}\frac{-1}{\sqrt{9 - x} + 3} = \frac{-1}{\sqrt{9 - x} + 3}$$

Now we have an expression for f as a quotient where the bottom does not have limit zero at 0. Apply the theorem on limits of quotients to get

$$\lim_{x \to 0} f(x) = \lim_{x \to 0} \frac{-1}{\sqrt{9 - x} + 3} = \frac{\lim_{x \to 0} (-1)}{\lim_{x \to 0}\left(\sqrt{9 - x} + 3\right)} = \frac{-1}{\sqrt{9} + 3}$$

§2.1 #22 Task a) Give an example of a real c and functions f and g such that

$$\text{neither } \lim_{x \to c} f(x) \text{ nor } \lim_{x \to c} g(x) \text{ exist, but } \lim_{x \to c} [f(x) + g(x)] \text{ does exist.}$$

b) Give an example of a real c and functions f and g such that

$$\text{neither } \lim_{x \to c} f(x) \text{ nor } \lim_{x \to c} g(x) \text{ exist, but } \lim_{x \to c} [f(x) \cdot g(x)] \text{ does exist.}$$

c) Give an example of a real c and functions f and g such that

$$\text{neither } \lim_{x \to c} f(x) \text{ nor } \lim_{x \to c} g(x) \text{ exist, but } \lim_{x \to c} [f(x) / g(x)] \text{ does exist.}$$

Results

a) Take $c = 0$, $f(x) = 1/x$, and $g(x) = -1/x$.

b) Take $c = 0$,

$$f(x) = \begin{cases} 1 & \text{if } x < 0 \\ -1 & \text{if } x > 0 \end{cases}, \text{ and } g(x) = f(x)$$

b) Take $c = 0$, f and g as in (b)

Verification

a) If $\lim_{x\to 0} f(x)$ exists, call it L. Since $\lim(1/n) = 0$ the sequential criterion for functional limits would force us to have $\lim_{x\to 0} f(x) = \lim(f(1/n)) = \lim(n) = L$. This is impossible since the sequence $(n)_{n=1}^{\infty}$ is unbounded. Similary, g cannot have a limit at 0. But $f + g$ is defined on the non-zero reals and has constant output 0. so $f + g$ has limit 0 at 0.

b) Both f and g have jumps at 0. So neither has a limit at 0. On the other hand fg is a constant function, with output 1 for all x in its domain. Thus fg does have a limit at 0.

c) As in (b) the functions f and g fail to have limits at 0. Here the quotient function f/g has constant output 1 on its domain, so it has a limit at 0.

Ch 2. #25 The solution is subtle. I will post an elegant solution eventually. I have asked the grader to look for reasonable attacks based on my solution in its current inelegant form.