Homework # 5 , Math 311:02, Fall 2008
Sample Solutions

§2.1 #2 TASK Set $D = (-2, 0)$ and define f on D by

$$f(x) = \frac{2x^2 + 3x - 2}{x + 2}$$

Show that f has a limit at -2, find the value of the limit, and prove the result.
EXPLORATION. We need to validate two statements

1. -2 is an accumulation point of D
2. There is a real L such that for every positive ε there is a positive δ such that

$$\text{for all } x \text{ in } D,\ 0 < |x - (-2)| < \delta \Rightarrow |f(x) - L| < \varepsilon$$

PROOF (1) Since $-2 = \lim \left(-2 + \frac{1}{n}\right)$, and all $-2 + \frac{1}{n}$ belong to $D - \{-2\}$, -2 is indeed an accumulation point of D.

(2) To find a likely L we first try to use the result on limits of quotients. We can't because the limit of the bottom is 0 as $x \to -2$. So we simplify the formula for f. Note that for x in $Dom(f)$, $x + 2 > 0$. Thus for x in $Dom(f)$

$$f(x) = \frac{2x^2 + 3x - 2}{x + 2} = \frac{(x + 2)(2x - 1)}{x + 2} = \frac{x + 2}{x + 2} \cdot (2x - 1) = 1 \cdot (2x - 1)$$

As $x \to -2$, $2x - 1 \to -4 - 1$. Conjecture that $L = -5$.

Consider an arbitrary positive ε. Keep x in D. Note that

$$\begin{aligned} |f(x) - L| &= |(2x - 1) - (-5)| = |2x + 4| = 2|x - (-2)| \quad \text{and so} \\ |f(x) - L| &< \varepsilon \Leftrightarrow 2|x - (-2)| < \varepsilon \Leftrightarrow |x - (-2)| < \varepsilon/2. \end{aligned}$$

Take $\delta = \varepsilon/2$. Suppose that $x \in D - \{-2\}$ and $0 < |x - (-2)| < \delta$. Then

$$|f(x) - L| = 2|x - (-2)| < 2\delta = \varepsilon.$$

§2.1 #5. Suppose that x_0 is an accumulation point of the domain of function f. Suppose also that

$$\lim_{x \to x_0} f(x) = L_1 \quad \text{and} \quad \lim_{x \to x_0} f(x) = L_2.$$

Show that $L_1 = L_2$.
EXPLORATION It is enough to show that for every positive ε, $|L_1 - L_2| < \varepsilon$.
PROOF Consider an arbitrary positive ε. Note that $\varepsilon/2 > 0$. By the two limit assumptions we get positive δ_1 and δ_2 such that

$$\begin{aligned} \text{for all } x \text{ in } D,\ 0 &< |x - x_0| < \delta_1 \Rightarrow |f(x) - L_1| < \varepsilon/2 \\ \text{for all } x \text{ in } D,\ 0 &< |x - x_0| < \delta_2 \Rightarrow |f(x) - L_2| < \varepsilon/2 \end{aligned}$$

Set $\delta = \min(\delta_1,\ \delta_2)$. Note $\delta > 0$. Since x_0 is an accumulation point of D, we get a real number s such that $s \in V_\delta(x_0) \cap D - \{x_0\}$. Thus

$$0 < |s - x_0| < \delta_1 \quad \text{and} \quad 0 < |s - x_0| < \delta_2.$$

So we get

$$|L_1 - L_2| = |L_1 - f(s) + f(s) - L_2| \le |L_1 - f(s)| + |f(s) - L_2| < \varepsilon/2 + \varepsilon/2 = \varepsilon.$$

§2.1 #8. TASK Set $D = (0, 1)$ and define f on D by

$$f(x) = \frac{x^3 - x^2 + x - 1}{x - 1}$$

Show that f has a limit at 1.
EXPLORATION We need to validate two statements

1. 1 is an accumulation point of D
2. There is a real L such that for every positive ε there is a positive δ such that

$$\text{for all } x \text{ in } D,\ 0 < |x - 1| < \delta \Rightarrow |f(x) - L| < \varepsilon$$

PROOF We have seen that endpoints of intervals are accumulation points. It remains to find an L that satisfies the second statement.

As in #2, we cannot use the limit-of-a-quotient theorem. Note that the top vanishes if we evaluate it at $x = 1$. That means that the top has $(x - 1)$ as a factor. Indeed, on D

$$f(x) = \frac{x^3 - x^2 + x - 1}{x - 1} = \frac{x^2(x-1) + (x-1)}{x - 1} = \frac{x - 1}{x - 1} \cdot \left(x^2 + 1\right) = x^2 + 1$$

since $(x - 1) / (x - 1) = 1$ for all x in D. We conjecture that the limit will be $L = 1^2 + 1 = 2$.

Consider an arbitrary positive ε. For all x in the domain of our function

$$|f(x) - 2| = \left|\left(x^2 + 1\right) - 2\right| = \left|x^2 - 1\right| = |x + 1| \cdot |x - 1|$$

Keeping $x \in D$ we get

$$\begin{aligned} 0 &< x < 1 \text{ and so } 1 < x + 1 < 2 \text{ and so} \\ |x + 1| &= x + 1 < 2 \text{ and finally } |f(x) - 2| = |x + 1| \cdot |x - 1| < 2 \cdot |x - 1|. \end{aligned}$$

Note that

$$2|x - 1| < \varepsilon \Leftrightarrow |x - 1| < \varepsilon/2$$

Choose $\delta = \varepsilon/2$.

Now suppose that $x \in D$ and $0 < |x - 1| < \delta$. Then

$$|f(x) - 2| = |x + 1| \cdot |x - 1| < 2 \cdot |x - 1| < 2 \cdot \varepsilon/2 = \varepsilon.$$

§2.1 #12 TASK Suppose that f is a function, $D = Dom(f)$, and f has limit L at x_0. Show that $|f|$ has limit $|L|$ at x_0.
EXPLORATION The hypothesis implicitly tells us that x_0 is an accumulation point of D. The domain of $|f|$ is the same as the domain of f. So all we need to do is verify the $\varepsilon\delta$-condition for $\lim_{x\to x_0} |f(x)| = |L|$.
PROOF Suppose ε is an arbitrary positive real. By hypothesis we get a δ such that

$$\text{for all } x \text{ in } D,\ 0 < |x - x_0| < \delta \Rightarrow |f(x) - L| < \varepsilon.$$

Use this δ in the proof for $|f|$.

Suppose that $x \in D$ and that $0 < |x - x_0| < \delta$. Recall that for all real u and v

$$|\,|u| - |v|\,| \leq |u - v|.$$

So now

$$|\,|f(x)| - |L|\,| \leq |f(x) - L| < \varepsilon$$

§2.1 #15. Consider a real-valued function f with $D = Dom(f) \subseteq \mathbb{R}$. Assume that x_0 is an accumulation point for D. Show that
IF f satisfies the condition

(C) for each positive ε there is a neighborhood Q of x_0 such that for all x and y in $Q \cap D - \{x_0\}$, $|f(x) - f(y)| < \varepsilon$.

THEN

$$\lim_{x\to x_0} f(x) \text{ exists.}$$

EXPLORATION Recall that

$$Q \text{ is a neighborhood of } x_0 \text{ iff there is a positive } r \text{ so that } V_r(x_0) \subseteq Q$$

and

$$V_r(x_0) = \{s : |s - x_0| < r\}.$$

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PROOF Assume that f satisfies condition (C). We proceed in two steps. First we find a likely value L for the limit. Then we verify that $\lim_{x\to x_0} f(x) = L$.

We might try $L = f(x_0)$ but we have no reason to believe that x_0 even belongs to D. Instead I will take a sequence of legal inputs for f converging to x_0 and show that the corresponding sequence of outputs converges. Then I will take L to be that limit of outputs.

By assumption x_0 is an accumulationpoint of D. That means we get a sequence $(x_n)_{n\in\mathbb{N}}$ in $D - \{x_0\}$ with $\lim(x_n) = x_0$. For each n, set $w_n = f(x_n)$. To show that $\lim(w_n)$ exists, it is enough to show that $(w_n)_{n\in\mathbb{N}}$ is Cauchy. So consider an arbitrary positive ε. Use (C) to get a neighborhood Q of x_0 with the property that

$$\text{for all } u, v \text{ in } Q \cap D - \{x_0\},\ |f(u) - f(v)| < \varepsilon.$$

Since this Q is a neighborhood of x_0 we get a positive r such that

$$V_r(x_0) \subseteq Q.$$

Since $\lim(x_n) = x_0$ and $r > 0$ we get N in $\mathbb{N}$ such that

$$\text{whenever } n \geq N, \text{ then } |x_n - x_0| < r.$$

Now keep both $m \geq N$ and $n \geq N$. Since we have

$$\begin{aligned} x_m &\in V_r(x_0) \cap D - \{x_0\} \subseteq Q \cap D - \{x_0\} \quad \text{and} \\ x_n &\in V_r(x_0) \cap D - \{x_0\} \subseteq Q \cap D - \{x_0\} \end{aligned}$$

we get

$$|w_m - w_n| = |f(x_m) - f(x_n)| < \varepsilon.$$

This shows that the sequence $(w_n)_{n\in\mathbb{N}}$ is Cauchy and thus convergent. Set $L = \lim(w_n) \in \mathbb{R}$.

Next I show that $\lim_{x\to x_0} f(x) = L$.

Consider an arbitrary positive ε. Note that $\varepsilon/2 > 0$. Use (C) to get a neighborhood Q of x_0 this time with the property that

$$\text{for all } u, v \text{ in } Q \cap D - \{x_0\}, \ \ |f(u) - f(v)| < \varepsilon/2.$$

Since Q is a neighborhood of x_0 we get positive r such that $V_r(x_0) \subseteq Q$. Set $\delta = r$. Use the sequence $(x_n)_{n\in\mathbb{N}}$ from the first step, converging to x_0. Get an N_1 so that

$$\text{whenever } n \geq N_1, \ |w_n - L| < \varepsilon/2.$$

Get an N_2 so that

$$\text{whenever } n \geq N_2, \ |x_n - x_0| < r.$$

Choose $N_3 = \max(N_1, \ N_2)$,

Now suppose that $x \in D$ and $0 < |x - x_0| < \delta = r$. We have

$$\begin{aligned} x &\in V_r(x_0) \cap D - \{x_0\} \subseteq Q \cap D - \{x_0\} \quad \text{and} \\ x_{N_3} &\in V_r(x_0) \cap D - \{x_0\} \subseteq Q \cap D - \{x_0\} \end{aligned}$$

Thus by (C)

$$\left|f(x) - f\left(x_{N_3}\right)\right| < \varepsilon/2$$

But also

$$\left|f\left(x_{N_3}\right) - L\right| = \left|w_{N_3} - L\right| < \varepsilon/2$$

So

$$|f(x) - L| \leq \left|f(x) - f\left(x_{N_3}\right)\right| + \left|f\left(x_{N_3}\right) - L\right| < \varepsilon/2 + \varepsilon/2 = \varepsilon$$

TASK §1.1 #7 Suppose that $(a_n)_1^\infty$ is a sequence in $\mathbb{R}$ and $A \in \mathbb{R}$. Show that

$$(a_n)_1^\infty \text{ converges to } A \quad \text{iff} \quad (a_n - A)_1^\infty \text{ converges to } 0.$$

PROOF It will be convenient to define, for each index n, $b_n = a_n - A$.

Step 1. Assume that $(a_n)_1^\infty$ converges to A and deduce that $(b_n)_1^\infty$ converges to 0.

Consider an arbitrary positive ε. We need to find an N with the property that

$$\text{whenever } n \geq N \text{ then also } |b_n - 0| < \varepsilon.$$

Note that for any index n, $|b_n - 0| < \varepsilon \iff |a_n - A| < \varepsilon$. By the convergence assumed for $(a_n)_1^\infty$ we get an N_a with the property that

$$\text{whenever } n \geq N_a \text{ then also } |a_n - A| < \varepsilon.$$

Thus it is enough to take $N = N_a$.

Step 2. Assume that $(b_n)_1^\infty$ converges to 0 and deduce that $(a_n)_1^\infty$ converges to A.

Consider an arbitrary positive ε. We need to find an N with the property that

$$\text{whenever } n \geq N \text{ then also } |a_n - A| < \varepsilon.$$

Note that for any index n, $|b_n - 0| < \varepsilon \iff |a_n - A| < \varepsilon$. By the convergence assumed for $(b_n)_1^\infty$ we get an N_b with the property that

$$\text{whenever } n \geq N_b \text{ then also } |b_n - 0| < \varepsilon.$$

Thus it is enough to take $N = N_b$.