Homework # 3 , Math 311:02, Fall 2008
Sample Solutions

TASK 1.1 #1 Show that $[0,1]$ is a neighborhood of $2/3$.
EXPLORATION. By deønition of the term ıneighborhoodȷ our task is to show that

$$\text{there exists a postive } \varepsilon \text{ such that } \left(\frac{2}{3}-\varepsilon, \frac{2}{3}+\varepsilon\right) \subseteq [0,1].$$

Such an ε much satisfy $0 \leq \frac{2}{3}-\varepsilon < \frac{2}{3}+\varepsilon \leq 1$Thus we much show there exists an ε satisfying both $\varepsilon \leq 2/3$ and $\varepsilon \leq 1-2/3 = 1/3$.
PROOF Set $\varepsilon = 1/3$. Note that $\left(\frac{2}{3}-\varepsilon, \frac{2}{3}+\varepsilon\right) = \left(\frac{1}{3}\ ,\ 1\right)$.Consider an arbitrary element x of $\left(\frac{1}{3}\ ,\ 1\right)$. We have $\frac{1}{3} < x < 1$. Thus we also have $0 \leq x \leq 1$, which means $x \in [0,1]$.

TASK 1.1 #3 Suppose that $x \in \mathbb{R}$ and $\varepsilon > 0$. Show that $(x-\varepsilon\ ,\ x+\varepsilon)$ is a neighborhood of each of its elements.
EXPLORATION We must show that for every element w of $(x-\varepsilon\ ,\ x+\varepsilon)$ there is a positive δ with the property that $(w-\delta\ ,\ w+\delta) \subseteq (x-\varepsilon\ ,\ x+\varepsilon)$.
PROOF Consider an arbitrary w in $(x-\varepsilon\ ,\ x+\varepsilon)$. Since w must be strictly between the endpoints of the given interval $(x-\varepsilon\ ,\ x+\varepsilon)$, the distance between w and the nearest endpoint must be positive. Set $\delta = \min\{w-(x-\varepsilon)\ ,\ (x+\varepsilon)-w\}$. It follows that

$$x-\varepsilon \leq w-\delta < w+\delta < x+\varepsilon$$

and thus that an arbitrary element z of $(w-\delta\ ,\ w+\delta)$ must belong to $(x-\varepsilon\ ,\ x+\varepsilon)$ since

$$x-\varepsilon \leq w-\delta < y < w+\delta < x+\varepsilon$$

NOTE We chose the largest δ that will meet our needs. Any smaller positive δ will do as well.

TASK 1.1 #6a,c. Use only the deønition of convergence to show that each of the following sequences converges

$$(a) \quad \left(5+\frac{1}{n}\right)_{n=1}^{\infty} \qquad\qquad (c) \quad \left(2^{-n}\right)_{n=1}^{\infty}$$

EXPLORATION Recall that $(s_n)_1^{\infty}$ converges iœ there is a real L such that

for each positive real ε there is a positive integer N such that whenever $n \geq N$ it follows that $|s_n - L| < \varepsilon$.

PROOF (a) Since the terms $1/n$ get very small when n gets very large we conjecture that the sequence in (a) converges to 5. We can choose $L = 5$. Consider an arbitrary positive ε. Note that for all index values n

$$\left|\left(5+\frac{1}{n}\right)-5\right| < \varepsilon \iff \frac{1}{n} < \varepsilon \iff \frac{1}{\varepsilon} < n$$

We now choose $N = \left\lfloor \frac{1}{\varepsilon} \right\rfloor + 1$, where I am using $\lfloor\rfloor$ to denote the "greatest integer" function. Now whenever we have $n \geq N$ it follows that

$$n > \frac{1}{\varepsilon} \text{ and thus } \left|\left(5+\frac{1}{n}\right)-5\right| < \varepsilon$$

PROOF (c) The ørst several terms in our sequence are

$$\frac{1}{2}, \ \frac{1}{4}, \ \frac{1}{8}, \ \ldots$$

so we take $L = 0$ and conjecture that the sequence converges to L. It is easy to prove by induction that for all positive integers n, $n \leq 2^n$.

Consider an arbitrary postive ε. Note that for all indices

$$\left|2^{-n} - 0\right| < \varepsilon \iff \frac{1}{2^n} < \varepsilon \iff \frac{1}{\varepsilon} < 2^n$$

It is not immediately obvious how to "solve" this last inequality for n. What we do instead is use the remark above that $n \leq 2^n$ for all positive integers. Pick $N = 1 + \lfloor 1/\varepsilon \rfloor$. Whenever $n \geq N$ we get

$$2^n \geq n > 1/\varepsilon \qquad \text{and thus} \qquad \left|2^{-n} - 0\right| < \varepsilon.$$

TASK 1.1 #4 Let $s = \left(\frac{3n+7}{n}\right)_{n\in\mathbb{N}}$. Find upper and lower bounds for the sequence s.

EXPLORATION We don't need the least upper bound or the greatest lower bound..

RESULT 10 is an upper bound for s; 0 is a lower bound.

PROOF For each n in $\mathbb{N}$ we have

$$0 < 3 < \frac{3n+7}{n} = 3 + \frac{7}{n} \leq 3 + 7 = 10$$

TASK 1.1 #9. Suppose we have three sequences of reals $(a_n)_1^\infty$, $(b_n)_1^\infty$, $(c_n)_1^\infty$ and a real number A with the properties that

$$(H1) \qquad (a_n)_1^\infty \text{ converges to } A \text{ and } (c_n)_1^\infty \text{ converges to } A$$

$$(H2) \qquad \text{for all indices } n,\ a_n \leq b_n \leq c_n.$$

Show that $(b_n)_1^\infty$ also converges to A.

EXPLORATION The idea is that for very large n both a_n and c_n are very close the this same number A. Since b_n is between a_n and c_n, b_n should be near A also..

PROOF Consider an arbitrary postive ε. We need to ønd a postive integer N such that whenever $n \geq N$ then also $|b_n - A| < \varepsilon$.

By $(H1)$ we get a positive integer N_a such that whenever $n \geq N_a$ then $|a_n - A| < \varepsilon$. By $(H1)$ we also get a positive integer N_c such that whenever $n \geq N_c$ then $|c_n - A| < \varepsilon$. Now by $(H2)$ we learn that if both $n \geq N_a$ and $n \geq N_c$ we get

$$A - \varepsilon \ < \ a_n \leq b_n \leq c_n < A + \varepsilon, \text{ which tells us that}$$
$$-\varepsilon \ < \ b_n - A < \varepsilon \ \text{ and ønally } \ |b_n - A| < \varepsilon.$$

So what positive integer should we take for N? We want N big enough that whenever $n \geq N$, then also $n \geq N_a$ and $n \geq N_c$. We can take $N = \max(N_a,\ N_c)$.

TASK 1.1 #7 Suppose that $(a_n)_1^\infty$ is a sequence in $\mathbb{R}$ and $A \in \mathbb{R}$. Show that

$$(a_n)_1^\infty \text{ converges to } A \qquad \text{iœ} \qquad (a_n - A)_1^\infty \text{ converges to } 0.$$

PROOF It will be convenient to deøne, for each index n, $b_n = a_n - A$.
Step 1. Assume that $(a_n)_1^\infty$ converges to A and deduce that $(b_n)_1^\infty$ converges to 0.

Consider an arbitrary positive ε. We need to ønd an N with the property that

$$\text{whenever } n \geq N \text{ then also } |b_n - 0| < \varepsilon.$$

Note that for any index n, $|b_n - 0| < \varepsilon \iff |a_n - A| < \varepsilon$. By the convergence assumed for $(a_n)_1^\infty$ we get an N_a with the property that

$$\text{whenever } n \geq N_a \text{ then also } |a_n - A| < \varepsilon.$$

Thus it is enough to take $N = N_a$.
Step 2. Assume that $(b_n)_1^\infty$ converges to 0 and deduce that $(a_n)_1^\infty$ converges to A.

Consider an arbitrary positive ε. We need to ønd an N with the property that

$$\text{whenever } n \geq N \text{ then also } |a_n - A| < \varepsilon.$$

Note that for any index n, $|b_n - 0| < \varepsilon \iff |a_n - A| < \varepsilon$. By the convergence assumed for $(b_n)_1^\infty$ we get an N_b with the property that

$$\text{whenever } n \geq N_b \text{ then also } |b_n - 0| < \varepsilon.$$

Thus it is enough to take $N = N_b$.