Homework #2, Math 311:02, Fall 2008
Sample Solutions

0.3 #24 TASK: Define a function f from $\mathbb{N}$ into $\mathbb{N}$ by

$$\begin{aligned} f(1) &= 1 \qquad f(2) = 2 \qquad f(3) = 3 \qquad \text{and} \\ \text{whenever } n &\geq 4,\ f(n) = f(n-1) + f(n-2) + f(n-3). \end{aligned}$$

Show that

$$\text{for all } n \text{ in } \mathbb{N},\ f(n) \leq 2^n$$

EXPLORATION We check the assertion for several values of n

$$\begin{aligned} \text{when } n &= 1,\ f(n) = f(1) = 1 \leq 2 = 2^1 = 2^n \\ \text{when } n &= 2,\ f(n) = f(2) = 2 \leq 4 = 2^2 = 2^n \\ \text{when } n &= 3,\ f(n) = f(3) = 3 \leq 8 = 2^3 = 2^n \\ \text{when } n &= 4,\ f(n) = f(4) = f(3) + f(2) + f(1) = 3 + 2 + 1 \leq 16 = 2^4 = 2^n \\ \text{when } n &= 5,\ f(n) = f(5) = f(4) + f(3) + f(2) = 6 + 3 + 2 \leq 32 = 2^5 = 2^n \end{aligned}$$

PROOF

For each n in $\mathbb{N}$, let $P(n)$ denote the assertion

$$\text{for all positive integers } k \text{ with } k \leq n,\ f(k) \leq 2^k.$$

We have already seen that $P(n)$ is true whenever $n \in \{1, 2, 3, 4\}$.

I now prove by induction that for all integers n with $n \geq 4$ that $P(n)$ is true.

The base case is now the case $n = 4$. $P(4)$ was proved in the exploration.

Suppose the $n \in \{k \text{ in } \mathbb{N} : k \geq 4\}$ and $P(n)$ is true. Note that since $n \geq 4$, $n-1 \in \mathbb{N}$ and $n-2 \in \mathbb{N}$ and $n-3 \in \mathbb{N}$. We will deduce that $P(n+1)$ is also true. By $P(n)$

$$f(n) \leq 2^n \qquad f(n-1) \leq 2^{n-1} \text{ and } f(n-2) \leq 2^{n-2}.$$

Thus

$$\begin{aligned} f(n+1) &= f(n) + f(n-1) + f(n-2) \\ &\leq 2^n + 2^{n-1} + 2^{n-2} = 2^{n-2}\left(2^2 + 2^1 + 1\right) \\ &\leq 2^{n-2}(7) < 2^{n-2} \cdot 8 = 2^{(n-2)+3} = 2^{n+1} \end{aligned}$$

By the induction hypothesis $P(n)$ we know that $f(k) \leq 2^k$ whenever $k \leq n$. We have just shown that $f(k) \leq 2^k$ whenever $k = n+1$. Thus $P(n+1)$ follows.

0.4 #32 TASK: Suppose that $n \in \mathbb{N}$. Let P_n denote the set of all polynomials of degree exactly n and integer coefficients. Show that P_n is countable.
EXPLORATION We will try to use the results of Section 0.4 to avoid doing hard work. So we know that

(Cor 0.15) any subset of a countable set is countable;

(Thm 0.16) the Cartesian product of two countable sets is countable, and thus by a simple induction the cartesian product of any finite number of countable sets is countable;

(Thm 0.17) a countable union of countable sets is countable.

PROOF A polynomial of degree n with integer coefficients is a function of the form

$$g(x) = \sum_{k=0}^{n} c_k \, x^k$$

where each $c_k \in \mathbb{Z}$ and $c_n \neq 0$. Thus there is a one-one function f from P_n onto $\mathbb{Z} \times \mathbb{Z} \times ... \times \mathbb{Z} \times (\mathbb{Z} - \{0\})$ where we have n copies of $\mathbb{Z}$. This f is given by

$$f\left(\sum_{k=0}^{n} c_k \, x^k\right) = (c_0, c_1, ..., c_n)$$

Two polynomials are equal if and only if their ordered strings of coefficients are equal. So this function f is indeed one to one. By definition of degree n the function f is onto. Now the Cartesian product of n copies of $\mathbb{Z}$ and one copy of $\mathbb{Z} - \{0\}$ is a product of a finite number of countable sets, so $P_n\tilde{}$ a countable set and is thus countable.

0.4 #38 TASK Suppose that $a < b$ and $c < d$. Show that $[a, b]\tilde{}[c, d]$.

REMARK The statement is not true in the generality used in the text. The interval $[0, 0]$ is certainly not equivalent to the interval $[0, 1]$ – the first contains one and only one element, namely 0; the second is infinite since it contains the subset $\{1/k : k \in \mathbb{N}\}$ which is not finite.

PROOF It is easy to construct a polynomial function of degree 1 that maps $[a, b]$ one to one onto $[c, d]$. The graph of this polynomial is the straight line segment with endpoints (a, c) and (b, d). Take

$$m = \frac{d-c}{b-a} \qquad \text{and} \qquad f(x) = b + m(x-a)$$

Since $m > 0$ it is easy to see that

$$\text{whenever } a \leq r < s \leq b \text{ then } c = f(a) \leq f(r) < f(s) \leq f(b) = d$$

and thus that f maps $[a, b]$ one-to-one into $[c, d]$. It remains to show that f is onto. Consider an arbitrary y in $[c, d]$. I need to show that there is an x in $[a, b]$ such that $f(x) = y$. Now for any real x

$$f(x) = y \Longleftrightarrow m(x-a) + c = y \iff \frac{y-c}{m} = x - a \iff x = a + \frac{y-c}{m}$$

We are done as soon as we see why $a + (y-c)/m \in [a, b]$. Since $y \leq [c, d]$ and $m > 0$ we get

$$c \ \leq \ y \leq d \text{ and so } 0 \leq \frac{y-c}{m} \leq \frac{d-c}{m} = b - a$$
$$\text{and so } a \ \leq \ a + \frac{y-c}{m} \leq b \text{ which means } a + \frac{y-c}{m} \in [a, b].$$

0.5 #41 TASK Suppose that $0 < a < b$. Show that $0 < a^2 < b^2$ and $0 < \sqrt{a} < \sqrt{b}$.

REMARK For this problem we will assume that every positive real r have a unique positive real square root denoted by $\sqrt{r}$.

PROOF
Step 1. Show that $0 < a^2$. This follows by the order axiom that says the product of positive reals is positive.
Step 2. Show that $a^2 < b^2$. By hypothesis, $b - a$ is positive. Now

$$b^2 = [a + (b-a)]^2 = a^2 + 2 \cdot a \cdot (b-a) + (b-a)^2$$

Note that both $2 \cdot a \cdot (b-a)$ and $(b-a)^2$ are positive since they are products of positive reals. Thus

$$2 \cdot a \cdot (b-a) + (b-a)^2 > 0$$

and

$$b^2 = a^2 + 2 \cdot a \cdot (b-a) + (b-a)^2 > a^2.$$

Step 3 Show that $0 < \sqrt{a} < \sqrt{b}$. By the meaning of $\sqrt{a}$ we know $\sqrt{a} > 0$. To get the second inequality we appeal to trichotomy.

Suppose $\sqrt{a} = \sqrt{b}$. Then

$$a = \left(\sqrt{a}\right)^2 = \left(\sqrt{b}\right)^2 = b, \text{ which is false.}$$

So we learn that $\sqrt{a} \neq \sqrt{b}$.

Suppose that $\sqrt{b} < \sqrt{a}$. Then by the argument of Step 2 we would learn that

$$b = \left(\sqrt{b}\right)^2 < \left(\sqrt{a}\right)^2 = a, \text{ which is false.}$$

So we learn that $\sqrt{b} \not< \sqrt{a}$.

We must conclude then that $\sqrt{a} < \sqrt{b}$.

0.5 #44 TASK Suppose that $x = lub(S)$. Show that for each positive ε there is an element s in S such that $x - \varepsilon < s \leq x$.

REMARK Implicit in the hypothesis are the assumptions that $\phi \neq S \subseteq \mathbb{R}$ and $x \in \mathbb{R}$.

PROOF Consider and arbitrary positive ε. Since $x = \min(\mathcal{UB}(S))$ and $x - \varepsilon < x$ we know that $x - \varepsilon$ is not an upper bound for S. Thus there must be an s with the two properties $s \in S$ and $x - \varepsilon < s$. Pick one such and call it s_o. Since $s_o \in S$, we also know that s_o has the property that $s_o \leq x$. Thus there is an element in S, namely s_o, such that $x - \varepsilon < s_o \leq x$.