Homework Sets # 10 , Math 311:02, Fall 2008
Sample Solutions

§4.1 #16 Task Define f on $[0,2]$ by $f(x) = \sqrt{2x - x^2}$. **(a)** Verify the hypotheses of Rolle's theorem and **(b)** find a c with $0 < c < 2$ and $f\,'(c) = 0$.
Proof (a): Set $r(s) = \sqrt{s}$ for non-negative s. Set $P(x) = 2x - x^2$ for real x. Note that $P(x) = x(2-x)$, which gives non-negative outputs when $x \in [0,2]$. Thus we see that P maps the domain of f into the domain of r and that $f = r \circ P$. Since f is the composition of two continuous functions f is continuous on its domain. Since r is differentiable on positive values of s and since P maps the open interval $(0,2)$ into the positive reals, the Chain Rule Theorem tells us that f is differentiable on $(0,2)$ where

$$f\,'(x) = r'(P(x))P'(x) = \frac{1}{2}\left(2x - x^2\right)^{-1/2}(2-2x).$$

Finally we note that

$$f(0) = \sqrt{0} = 0 \text{ and } f(2) = \sqrt{0} = 0 \text{ so } f(0) = f(2).$$

In summary (i) f is continous on $[0,2]$; (ii) f is differentiable on $(0,2)$; and (iii) $f(0) = f(2)$. These are the three hypotheses for Rolle's theorem for f on $[0,2]$.
Result (b) $c = 1$
Work for (b) Note that for $0 < c < 2$

$$\begin{aligned} f\,'(c) &= 0 \iff \frac{1}{2}\left(2c - c^2\right)^{-1/2}(2-2c) = 0 \\ &\Leftrightarrow \frac{1}{2}\frac{2-2c}{\sqrt{2c - c^2}} = 0 \iff 2 = 2c \iff c = 1. \end{aligned}$$

§4.1 #17 Task Define f on $\mathbb{R}$ by $f(x) = 1/\left(1 + x^2\right)$. Show that the range of f has a maximum value and find the input(s) that produce that maximum output.
Result $\max(\{f(x) : x \in \mathbb{R}\}) = 1$. The only input to produce maximum output is 0.
Proof By the quotient rule we can show that, for all x,

$$f\,'(x) = -\frac{2x}{\left(1+x^2\right)^2}.$$

Since this derivative is non-negative on $(-\infty, 0]$ the function is increasing on that interval. In particular, whenever $x \leq 0$, we have $f(x) \leq f(0) = 1$. Since this derivative is non-positive on $[0, +\infty)$ the function is decreasing on $[0. + \infty)$. In particular, whenever $0 \leq x$ we have $f(x) \leq f(0) = 1$. So for all real x, $f(x) \leq f(0) = 1$. Note also that whenever $x \neq 0$, $1 + x^2 > 1$ and $f(x) = 1/(1+x^2) < 1$.

§4.1 #19 Task Show that the equation

$$\cos(x) = x^3 + x^2 + 4x$$

has exactly one solution in $[0, \pi/2]$.
Proof Consider the function defined on $[0, \pi/2]$ by

$$f(x) = \left(x^3 + x^2 + 4x\right) - \cos(x).$$

Our equation has solution at x if and only if $f(x) = 0$.

Compute the derivative of f :

$$f\,'(x) = \left(3x^2 + 2x + 4\right) - (-\sin(x)) = \left(3x^2 + 2x + 4\right) + \sin(x).$$

In the interval $[0, \pi/2]$ we note that

$$3x^2 \geq 0 \qquad 2x \geq 0 \qquad 4 > 0 \qquad \sin(x) \geq 0$$

so $f'(x) \geq 4 > 0$. The fact that the derivative is strictly positive in the interval $[0, \pi/2]$ means that the function is strictly increasing there and must be one-to-one there.

Note also that

$$\begin{aligned} f(0) &= 0 - \cos(0) = -1 < 0 \\ f(\pi/2) &= (\pi/2)^3 + (\pi/2)^2 + 4 \cdot (\pi/2) - \cos(\pi/2) = (\pi/2)^3 + (\pi/2)^2 + 4 \cdot (\pi/2) > 0 \end{aligned}$$

since $\cos(\pi/2) = 0$.

The intermediate value theorem tells us that $f(x) = 0$ must have at least one solution between 0 and $\pi/2$. The fact that f is strictly increasing between 0 and $\pi/2$ tells us that there cannot be more than one such solution.

§4.1 #20 Task Suppose that f maps $[0, 2]$ into $\mathbb{R}$, that f is differentiable, and that

$$f(0) = 0 \qquad f(1) = 2 \qquad f(2) = 2.$$

Show that

(i) there is a c_1 where $f'(c_1) = 0$

(ii) there is a c_2 where $f'(c_2) = 2$

(iii) there is a c_3 where $f'(c_3) = 3/2$.

Proof

(i) Apply Rolle's Theorem to f on $[1, 2]$. There is at least one c such that $1 < c < 2$ and $f'(c) = 0$. Pick one and call it c_1.

(ii) Apply the Mean Value Theorem to f on $[0, 1]$ There is at least one c such that $0 < c < 1$ and $f'(c) = (f(1) - f(0))/(1 - 0) = (2 - 0)/1 = 2$. Pick one and call it c_2.

(iii) The derivative f' has the intermediate value property on $[c_2, c_1]$. We have seen that

$$f'(c_2) = 2 \text{ and } f'(c_1) = 0.$$

We note that 3/2 is between 0 and 2. Thus there must be a c between c_1 and c_2 such that $f'(c) = 3/2$. Pick one and call it c_3.

§4.1 #28 Task Consider the function defined by $f(x) = 2x^3 + 3x^2 - 36x + 5$ on $[-1, 1]$.

(a) Show that f is one-to-one on $[-1, 1]$.

(b) Is f increasing or decreasing?

Result f is strictly decreasing and thus is one-to-one.

Proof Compute the derivative

$$f'(x) = 6x^2 + 6x - 36 = 6\left(x^2 + x - 6\right) = 6(x + 3)(x - 2)$$

and note that on our interval

$$x + 3 \geq -1 + 3 > 0 \qquad \text{and} \qquad x - 2 \leq 1 - 2 < 0.$$

So our derivative is always negative in $[-1, 1]$ and our function is strictly decreasing and thus one-to-one.

§4.1 #29 Task Show that the function $f(x) = x^3 - 3x^2 + 17$ is not one-to-one on $[-1, 1]$

Proof Compute the derivative

$$f'(x) = 3x^2 - 6x = 3x(x - 2)$$

This derivative is positive when $-1 \leq x < 0$ and negative when $0 < x \leq 1$.

Note that $f(-1) = 13 \qquad f(0) = 17 \qquad f(1) = 15$.

As x increases from -1 to 0, the function outputs increase from 13 to 17.

As x increases from 0 to 1, the function outputs decrease from 17 to 15.

Thus the function has output 15 twice, once for an input between -1 and 0 and once at imput 1. This is sufficient to show that f is not one-to-one.