Homework #1, Math 311:02, Fall 2008

Sample Solutions

0.1#6 TASK Suppose that A, B, C are sets.

(a) Prove that

$$\text{If } A \subseteq B, \text{ then } C - B \subseteq C - A.$$

(b) Either prove the converse or provide a counterexample.

PROOF

(a) Assume that $A \subseteq B$. [Note: Our book uses $\subset$ for the weak inclusion. I am using $\subseteq$ to emphasize that we are not limiting ourselves to the strong inclusion, the one that rules out equality.] I must show that

$$\text{for all } x,\ x \in C - B \Longrightarrow x \in C - A.$$

So consider an arbitrary x and assume $x \in C - B$. This means $x \in C$ and $x \notin B$. Since $A \subseteq B$ and $x \notin B$, we learn that $x \notin A$. Thus $x \in C - A$. $\square$

(b) The converse says

$$\text{If } C - B \subseteq C - A, \text{ then } A \subseteq B.$$

This need not be true. Consider the example

$$A = \{1\}\,,\ B = C = \phi$$

Then we have

$$C - A = \phi \text{ and } C - B = \phi, \text{ so } \ C - B \subseteq C - A \ \text{ but } A \subsetneq B$$

0.1#10 TASK b. Give a concise description of the set $B \doteqdot \bigcup_{n=1}^{\infty} (-n,n)$.

RESULT $B = \mathbb{R}$

REASONING By definition

$$B = \{x : \exists n \text{ in } \mathbb{N},\ x \in (-n,n)\}$$

B is a union of subsets of $\mathbb{R}$, so B is a subset of $\mathbb{R}$. To show the reverse inclusion, consider an arbitrary real r. We can find a positive integer n_r such that $-n_r < r < n_r$. So r belongs to the interval $(-n_r, n_r)$. Thus there is an index n, namely $n = n_r$, such that $r \in (-n,n)$. This shows that $r \in B$. $\square$

TASK d. Give a concise description of the set $D \doteqdot \bigcup_{n=1}^{\infty} \left(-\frac{1}{n}\ ,\ 2+\frac{1}{n}\right)$.

RESULT

$$D = \left(-\frac{1}{1}\ ,\ 2+\frac{1}{1}\right) = (-1,3)$$

REASONING By definition

$$D = \left\{x\ :\ \exists n,\ x \in \left(-\frac{1}{n}\ ,\ 2+\frac{1}{n}\right)\right\}.$$

Suppose that $x \in D$. Then we can and do pick a positive integer, call it n_o,

$$x \in \left(-\frac{1}{n_o}\ ,\ 2+\frac{1}{n_0}\right)$$

Since $n_o \geq 1$, we get

$$\frac{1}{n_o} \leq 1 \quad \text{so} \quad -1 \leq -\frac{1}{n_o} < x < 2+\frac{1}{n_o} < 2+1 \quad \text{and } x \in (-1,3)\,.$$

This shows that

$$D \subseteq (-1,3)\,.$$

Now suppose that $x \in (-1,3)$. Then with $n = n_1 = 1$ we get

$$x \in (-1,3) = \left(-\frac{1}{n_1}\ ,\ 2+\frac{1}{n_1}\right) \quad \text{so} \quad x \in D.$$

0.3 #20 TASK Prove that

$$\forall n \text{ in } \mathbb{N}, \quad 1 + 3 + ... + (2n - 1) = n^2$$

PROOF This calls for a proof by induction. For each positive integer n, let $P(n)$ denote the assertion

The sum of the first n odd integers is n^2

Base Step. Consider $n = 1$. The sum of the first n odd integers is just 1. Also $n^2 = 1$. So the assertion $P(1)$ is true.

Induction Step. Suppose that n is an arbitrary positive integer and that $P(n)$ is true. We must deduce the truth of $P(n + 1)$.

The sum of the first $n+1$ odd integers is the sum of the first n odd integers plus the $(n+1)^{st}$. This sum is, by the induction hypothesis just $n^2+(2n + 1)$. But

$$n^2 + (2n + 1) = (n + 1)^2$$

and this gives us

The sum of the first $n + 1$ odd integers is $(n + 1)^2$

which is exactly $P(n + 1)$.