Bolzano-Weierstrass Theorem (Alternate Version)

Thm. Every bounded sequence has at least one convergent subsequence.

Pf. Suppose that $(x_n)_{n\in\mathbb{N}}$ is a bounded sequence of real numbers. We will construct a strictly monotone increasing sequence $(n_k)_{k\in\mathbb{N}}$ in $\mathbb{N}$ such that the subsequence $(x_{n_k})_{k\in\mathbb{N}}$ converges. This construction is fairly complicated. First I lay out the main steps.

Step 1. Construct a nested sequence of close bounded intervals $([\ell_k\ ,\ r_k])_{k\in\mathbb{N}}$ and a related sequence of unbounded subsets $(\mathcal{N}_k)_{k\in\mathbb{N}}$ of $\mathbb{N}$ with the properties that for each k,

$$(r_{k+1}-\ell_{k+1}) = \frac{1}{2}\cdot(r_k-\ell_k) \quad (1)$$

$$\mathcal{N}_{k+1} = \{n \text{ in } \mathcal{N}_k\ :\ x_n\in[\ell_{k+1}\ ,\ r_{k+1}]\} \quad (2)$$

Step 2. Find a strictly increasing sequence of positive integers such that

$$\text{for all } k,\ n_k\in\mathcal{N}_k$$

Step 3. Show that $(x_{n_k})_{k\in\mathbb{N}}$ converges.

Carry out step 1.

Since our sequence is bounded we get a positive real M such that, for all n, $|x_n|\le M$.

We start with the bounded interval $I_0=[\ell_0,r_0]=[-M,M]$ and the unbounded set $\mathcal{N}_0=\mathbb{N}$. We bisect the interval I_0 to get

$$L_0 = [\ell_0,m_0] = \text{left "half" of } I_0$$

$$R_0 = [m_0,r_0] = \text{right "half" of } I_0$$

We look at two index sets

$$\mathcal{L}_0=\{n \text{ in } N_0 : x_n\in L_0\} \quad\text{and}\quad \mathcal{R}_k=\{n \text{ in } N_0 : x_n\in R_0\}$$

The union of these two index sets is all of N_0. Since the union of two bounded sets is bounded, and since N_0 is not bounded, one of these index sets is unbounded. Set

$$I_1 = [\ell_1,r_1]=\begin{cases} L_0 & \text{if } \mathcal{L}_0 \text{ is unbounded}\\ R_0 & \text{if } \mathcal{L}_0 \text{ is bounded (and thus } \mathcal{R}_0 \text{is unbounded)}\end{cases}$$

$$N_1 = \begin{cases}\mathcal{L}_0 & \text{if} \quad I_1=L_0\\ \mathcal{R}_0 & \text{if} \quad I_1=R_0\end{cases}.$$

Note that I_1 is one of the "halves" of I_0 so that

$$I_1 \subseteq I_0 \quad \text{and}$$

$$\text{length of } I_1 = \frac{1}{2}\cdot\text{length of } I_0=\frac{1}{2}\cdot 2M=M$$

Also note that we chose $I_1$to insure that there would be inønitely many index values n with $x_n\in I_1$.deøned $\mathcal{N}_1$ exactly so that $\mathcal{N}_1=\{n \text{ in } \mathcal{N}_0 : x_n\in I_1\}$ and so that $\mathcal{N}_1$ is unbounded.

We now make an inductive deønition for the rest of the intervals and index sets. Suppose $k \in \mathbb{N}$ and we have chosen interval $I_k = [\ell_k, r_k]$ and index set $\mathcal{N}_k$ so that

$$I_k \subseteq I_{k-1} \qquad \mathcal{N}_k = \{n : x_n \in I_k\} \qquad \text{and} \qquad \mathcal{N}_k \text{ is not bounded above}$$

Set m_k =midpoint of $I_k = (\ell_k + r_k)/2$. This midpoint splits I_k into two halves:

$$L_k = [\ell_k, m_k] \quad \text{and} \quad R_k = [m_k, r_k]$$

Consider the two index sets

$$\mathcal{L}_k = \{n \text{ in } \mathcal{N}_k : x_n \in L_k\} \quad \text{and} \quad \mathcal{R}_k = \{n \text{ in } \mathcal{N}_k : x_n \in R_k\}$$

Since $\mathcal{L}_k \cup \mathcal{R}_k = \mathcal{N}_k$ and $\mathcal{N}_k$ is unbounded, we know that if $\mathcal{L}_k$ is bounded then $\mathcal{R}_k$ is not bounded. Set

$$\begin{aligned} I_{k+1} &= [\ell_{k+1}, r_{k+1}] = \begin{cases} L_k & \text{if } \mathcal{L}_k \text{ is unbounded} \\ R_k & \text{if } \mathcal{L}_k \text{ is bounded (and thus } \mathcal{R}_k \text{ is unbounded)} \end{cases} \\ \mathcal{N}_{k+1} &= \begin{cases} \mathcal{L}_k & \text{if} \quad I_{k+1} = L_k \\ \mathcal{R}_k & \text{if} \quad I_{k+1} = R_k \end{cases} . \end{aligned}$$

Since I_{k+1} is a "half" of I_k we get

$$r_{k+1} - \ell_{k+1} = length\ of\ I_{k+1} = \frac{1}{2} \cdot length\ of\ I_k = \frac{1}{2} \cdot (r_k - \ell_k).$$

Note that I_{k+1} was chosen so that I_{k+1} contains x_n for inønitely many index values in $\mathcal{N}_k$. Further, $\mathcal{N}_{k+1}$ is exactly that unbounded set of index values for which $x_n \in I_{k+1}$.

This completes the use of the bisection trick for Step 1.

Work for Step 2.

Since N_1 is unbounded, it must be be non-empty. Since $\phi \neq \mathcal{N}_1 \subseteq \mathbb{N}$, we know that $\mathcal{N}_1$ contains a smallest element. Write

$$n_1 = \min(\mathcal{N}_1).$$

We deøne the positive integers n_k with $k > 1$ inductively. Since $\mathcal{N}_2$ is not bounded above, it must contain at least one element larger than n_1. Set

$$n_2 = \min(\{n \text{ in } \mathcal{N}_2 : n > n_1\})$$

Suppose $k \in \mathbb{N}$ and we have already picked n_k. Since $\mathcal{N}_{k+1}$ is not bounded above, $\mathcal{N}_{k+1}$ must contain at least one element larger than n_k. Set

$$n_{k+1} = \min(\{n \text{ in } \mathcal{N}_{k+1} : n > n_k\}).$$

We now have a strictly increasing sequence $(n_k)_{k \in \mathbb{N}}$.

Work for Step 3.

Since each $n_k \in \mathcal{N}_k$, we know that $x_{x_n} \in I_k = [\ell_k, r_k]$ or equivalently $\ell_k \leq x_{n_k} \leq r_k$.

The intervals I_k were chosen to be nested. Applying the Nested Intervals Theorem, we know that

$$\lim(\ell_k) = lub(\ell_k) = \ell_* \leq r_* = glb(\ell_k) = \lim(\ell_k)$$

We can check easily by induction that for all indices k

$$0 \le r_k - \ell_k = \left(\frac{1}{2}\right)^k (r_0 - \ell_0) = \left(\frac{1}{2}\right)^k \cdot 2M = 2^{-k} \cdot 2M$$

We already know that $\lim \left(2^{-k}\right) = 0$. By the remark at the end of the proof of the Nested Intervals Theorem, we ønd that $\ell_* = r_*$. Consider an arbitrary index k.

$$\ell_k \le x_{n_k} \le r_k \quad \text{and} \quad \ell_k \le \ell_* = r_* \le r_k$$

This means that both x_{n_k} and ℓ_* are in the interval $[\ell_k, r_k]$. So the distance between x_{n_k} and ℓ_* cannot be greater than the length of that interval

$$|x_{n_k} - \ell_*| \le 2M \cdot 2^{-k}$$

Applying the Squeeze theorem, we know that

$$\lim \left(x_{n_k}\right) = \ell_*.$$

□

Remark: At some (possibly all) substeps in Step 1, we might have had two half intervals containing x_n for inønitely many values of n. If we make a diœerent choice of subinterval, we would have obtained a diœerent monotone sequence (n_k) and thus could get a diœerent convergent subsequence.

The reason for the weasel word "could", is that there are weird examples. Think of the simple sequence $(1, 1, 1, ...)$ Where each entry is 1. We can get lots of diœerent strictly increasing sequences (n_k). But every one of them gives the same subsequence $(1, 1, 1, ...)$ since all $x_{n_k} = 1$.

Example: Consider the sequence $(-1, 1, -1, 1, ...)$ where $a_n = 1$ if n is even and $a_n = -1$ if n is odd. In our construction above, $M = 1$. When we split $[-M, M]$ in half at the midpoint, we ønd that $\{n : a_n \in [-M, 0]\} = \{1, 3, 5, ...\}$ and N $\{n : a_n \in [0, M]\} = \{2, 4, 6, ...\}$. Thus in our construction we picked $I_1 = [-M, 0]$, $\mathcal{N}_1 = 1, 3, 5, ...\}$ and $n_1 = 1$. Splitting I_1 in half at its midpoint, we ønd that

$$\begin{aligned} \{n : a_n \in [-M, -M/2]\} &= \{1, 3, 5, ...\} \\ \{n : a_n \in [-M/2, 0]\} &= \phi \\ n_2 &= 3 \end{aligned}$$

and indeed with repeated splits we get a_n in the left half for all odd indices and no a_n in the right halves. So our (n_k) sequence turns out to be $(1, 3, 5, ...)$ and our convergent subsequence turns out to be $(-1, -1, -1, ...)$.

Had we decided to use $I_1 = [0, M]$ then at every further split we would have gotten nothing in the left half and everything in the right half. This would have given us the diœerent sequence $(n_k) = (2, 4, 6, ...)$ and the diœerent convergent subsequence $(a_{2k}) = (1, 1, 1, ...)$.