VARIATIONS AT INFINITY IN CONTACT FORM GEOMETRY

A. Bahri

Dedicated to Felix Browder on his eightieth birthday

Some thirty years ago, we started a direction of research which involved the study of a variational problem given by the action functional on a space of Legendrian curves "dual" to a given contact form α. α is given on a three dimensional manifold M and has an associated Reeb vector-field ξ.

In order to define this variational problem, we chose a vector field v in $ker\alpha$ and we introduced the form $\beta = d\alpha(v, .)$. We then considered the functional $I(x) = \int_0^1 \alpha_x(\dot{x})dx$ on the space of curves $C_\beta = \{x \in H^1(S^1, M)\, s.t.\, \beta_x(\dot{x}) \equiv 0, \alpha(\dot{x}) = \text{positive constant}\}$[1]. After observing that the critical points of I of positive critical value were the periodic orbits of ξ, we tried to prove that the Palais-Smale condition or variants of it held for (I, C_β).

We very rapidly reached the conclusion that this variational problem behaved very differently from other, more classical problems in that its variations implied the existence of asymptotes and we were struck by the difficulties that we encountered as we were trying to understand the behavior of these asymptotes and to compute their contribution in our variations [0],[1].

Slowly, we understood them better; we found a natural geometric semi-flow for the related variations [2] and we moved on to define a related homology [3].

More recently, we were able to make some substantial progress in the computation of this homology [4], as well as in putting strong restrictions on the asymptotes that it involves [5].

Our work involves some assumptions that we set up as we were defining this line of research and finding our intermediate conclusions. We found along this process these assumptions to be reasonable.

We have established most of them by now, but for a few hypotheses which we state here:

First, we have assumed that v in $ker\alpha$ is non-singular.

Second, we have assumed that $\beta = d\alpha(v, .)$ is a contact form with the same orientation than α. Although we have shown in [6] how to modify the functional when this assumption is not satisfied, it is still easier to work under such an assumption and we thus would like, in the best of the worlds, to reduce the region of M where $\beta \wedge d\beta \lneq 0$ to be as small as possible.

Third, we have assumed in our work that the variational problem at infinity on the space $\cup\Gamma_{2m}, m \in N$, that is in the space of curves made of m pieces of ξ-orbits alternating with m pieces of $\pm v$-orbits, satisfied the Palais-Smale condition. We elaborate more below about the definition of our variational problem at infinity.

Finally, we have a set of independent conditions that warrant compactness for the (semi)flow-lines involved in our homology. This result is in fact not essential in the definition of this homology, neither is it (to a certain extent) essential in its computation [4]. It is essential though in the understanding of our variational problem. We will not be considering this set of assumptions here.

We will however establish two important results that are relevant to the second and third assumptions. In order to describe these results, we pause to define the variational problem at infinity:

On $\cup\Gamma_{2k}$, in fact on C_β, we can consider in fact not one, but two functionals which coincide when $M = \{x, (\beta \wedge d\beta)_x \gneq 0\}$.

Namely, the first functional on C_β is $I(x) = \int_0^1 \alpha_x(\dot{x})dx$ and the corresponding functional at infinity is $I_\infty(x) = \Sigma a_i$, where a_i is the length along ξ of the $i^{th}\xi$-piece.

[1] This set-up work was completed in collaboration with D.Bennequin.

The second functional has been introduced in [6] to take into account the fact that there might be regions in M where $\beta\wedge d\beta \lneq 0$. Then the appropriate functional is or seems to be

$$J(x) = \int_{\Sigma^+} \alpha_x(\dot{x}) + \frac{1}{1+\int_{\Sigma^-}\alpha_x(\dot{x})}$$

where $\Sigma^+ = \{x, (\beta\wedge d\beta)_x \gneq 0\}$, $\Sigma^- = \{x, (\beta\wedge d\beta)_x \lneq 0\}$ and the corresponding functional at infinity is

$$J_\infty(x) = \Sigma a_i t_i^+ + \frac{1}{1+\Sigma a_i t_i^-}$$

Here a_i is as above and $t_i^\pm$ is the time spent by the $i^{th}\xi$-piece in $\Sigma^\pm$ respectively.

We will in our statements and proofs below restrict to I_∞ on $\cup\Gamma_{2k}$.In view of [6], we should be considering J_∞. However, the results on I_∞, to the least in the framework of the present paper, but probably more generally, will translate immediately into results on J_∞, as we will see later.

Our first result in this paper is a quite interesting result, which is also quite particular since it is relevant to the standard contact structure α_0 on S^3. Namely, we establish:

Theorem 1. *There is a non singular vector field v in the kernel of the standard contact structure α_0 on S^3 such that:*

i) v is also in the kernel of a codimension one non singular foliation transverse to α_0

ii) There exists a smooth function $\lambda : S^3 \longrightarrow R^+ - \{0\}$, such that $d(\lambda\alpha_0)(v,.) = \beta$ is a contact form with the same orientation than α_0.

iii) v is a Morse-Smale vector field and has two periodic orbits; the first one is denoted O_1 and is attractive, the second one is denoted O_2 and is repulsive.

iv) The variational problem at infinity J_∞ (equal here to I_∞) satisfies the Palais-Smale condition on the flow-lines of a pseudo-gradient. In addition, its only critical points are curves of $\cup\Gamma_{2k}$ with all their ξ-pieces characteristic (that is v is mapped onto a vector collinear to v in the ξ-transport from one edge to the next edge of every ξ-piece of this critical point at infinity).

This result is quite interesting for three reasons: first, it states that there is a codimension one, non singular foliation γ transverse to the standard contact structure on S^3. Second, even though γ is a foliation transverse to α_0, with v spanning the intersection of the two kernels, $\beta = d\alpha(v,.)$ rotates in the same direction than α_0 along v, thus the condition $\frac{\beta\wedge d\beta}{\alpha\wedge d\alpha} \gneq 0$ is not so unreasonable, even under such circumstances. Third, this result indicates that, under minimal conditions on v, compactness should hold for I_∞, J_∞ on $\cup\Gamma_{2m}$.

We thus consider next the case of a more general contact structure $ker\alpha$ on a three dimensional manifold M^3 and we suppose that we have found a Morse-Smale nowhere zero vector-field v in its kernel. We then prove that, with such a vector field v, we can find a positive smooth function λ (λ is subject to a small number of constraints and is otherwise quite general) such that, with $\beta = d(\lambda\alpha)(v,.)$:

Theorem 2.

i)The region Σ^- of M can be localized in tiny hyperbolic neighborhoods of some special hyperbolic orbits of v.

ii) The variational problem I_∞ for $\lambda\alpha$ satisfies the Palais-Smale condition on the flow-lines of a pseudo-gradient.

In view of i) of Theorem 1, ii) of Theorem 1 in fact readily implies that J_∞ also satisfies the Palais-Smale condition on flow-lines (see section 4).

The proof of the theorems stated above contains more than their statements as it clearly indicates that the assumption that v is Morse-Smale is not essential. The assumption that seems to matter (see the proofs) is an assumption on Σ^-, the set where $\beta\wedge d\beta$ is negative: of course, this set depends on the contact form α. However, it has some "intrinsic part", some part where $\beta\wedge d\beta$ is forced to be negative. This part typically corresponds to the subset S of M where v is an "Anosov" vector field, that is carries with itself a hyperbolic structure and where $ker\alpha$ almost defines a foliation, that is $ker\alpha$ rotates very little in the v-transport. Typically, this is a Cantor set, e.g it is provided by a horse-shoe map [7], [8], and ξ keeps an almost constant direction between the stable and unstable foliations defined by the hyperbolic structure of S. Assuming that ξ is "transverse" to S, in a sense that remains to be defined since this can be a quite complicated set, we can build then a neighborhood of S by taking $\cup\phi_s(S), s \in (0,\epsilon)$, ϕ_s is the one parameter group of ξ.

Under such an assumption, we would then need that there are no sizable ξ-pieces of trajectories inside this neighborhood. Then, the critical points at infinity of J_∞ will essentially converge, as ϵ tends to zero, to critical points at infinity of I_∞ and statements of compactness on the latter functional will translate into statements of the former functional.

1. A foliation transverse to the standard contact structure on S^3 (Proof of Theorem 1).

Let $D^2 \times S^1$ be the standard solid torus, which we may view in R^3 as an infinite cylinder, with periodicity 2π in the z-direction. The equation of this solid torus reads then $x^2+y^2 \leq 1$.

Let $\alpha_0 = (y-x)dz + dx$ be a contact form on the solid torus tangent to its chore. α_0 is transverse to the boundary of the solid torus.

Gluing up two copies of this solid torus appropriately (the orbits of $\frac{\partial}{\partial z}$of one copy along the orbits of $-y\frac{\partial}{\partial x} + x\frac{\partial}{\partial y}$ for the other copy), we find the manifold S^3.

The traces of α_0 on the two copies of $\partial(D^2 \times S^1)$ do not glue up. We need to modify this construction so that we can then derive in this way on S^3 its standard contact structure.

This is completed through a monotone rotation around the z-axis over $[0, 2\pi]$, with angular rotation 1. x becomes then $x\cos z - y \sin z$, y becomes $x \sin z + y\cos z$, z remains z. α_0 is modified correspondingly.

We complete this modification on the two copies, the horizontal and the vertical ones, along the x-axis for the horizontal one $y^2+z^2 \leq 1$ and along the z-axis for the vertical one.

The traces of the corresponding α_0s' on the boundaries of the corresponding tori glue up then naturally and, after identifying the exiting normal, which we pick in $\ker\alpha_0$, along the boundary of one torus, with an entering normal, again in $ker\alpha_0$, along the boundary of the other torus, we derive a global contact structure $ker\alpha_0$.

On each solid torus, there is a non-singular foliation γ transverse to $ker\alpha_0$ and the boundary $S^1\times S^1$ of each torus is a compact leaf for this foliation. γ is given explicitly, for example as $xdx + ydy + (1-x^2-y^2)dz$. $ker\gamma \cap ker\alpha_0$ defines a vector-field v. The orbits of v along $S^1 \times S^1$ are precisely the traces that we have used in the glueing (after the appropriate rotation has been completed).

Let us compute on the explicit models $\alpha_0 \wedge d\alpha_0$ and $\beta\wedge d\beta$, where $\beta = d\alpha_0(v,.)$.

We find:

$$\alpha_0 \wedge d\alpha_0 = dx \wedge dy \wedge dz$$

For $\beta \wedge d\beta$, we need first to compute v, which we find to be:

$$v = -y(y-x)\frac{\partial}{\partial x} + (x(y-x)+x^2+y^2-1)\frac{\partial}{\partial y} + y\frac{\partial}{\partial z}$$

so that

$$\beta = d\alpha_0(v,.) = (dy-dx)\wedge dz(v,.) = -y(dy-dx) + ((x+y)(y-x)+x^2+y^2-1)dz$$

and

$$\beta\wedge d\beta = (2y^2+1)dx\wedge dy\wedge dz$$

Thus β is a contact form with the same orientation than α_0. It does not follow however that such a global β, after the gluing up has been completed, is available.

Indeed, over the gluing up process, there are two tori, two forms α_0 etc, one on each side. It is unclear that α_0 on one side will glue, as a form, with α_0 on the other side. Rather, it will glue with $\lambda\alpha_0$, where λ is a positive function, not necessarily equal to 1.

In order to glue up the two "convex" forms (relative to v) α_0 and $\lambda\alpha_0$ into a "convex" form, we need the help of a technical lemma which we have established and used in [9].

This lemma reads as follows:

Let α_1, ξ_1 be a contact form and its contact vector-field. Let v be a vector-field in $ker\alpha_1$.Let λ be a positive function and $\alpha = \lambda\alpha_1$ be another contact form defined with the Hamiltonian λ. Let ξ be the contact vector-field of α. We then have:

Lemma 1 [9]. $-d\alpha(v,[\xi,v]) = \lambda[(\frac{1}{\lambda}+(\frac{1}{\lambda})_{vv})(-d\alpha_1(v,[\xi_1,v]))+(\frac{1}{\lambda})_v d\alpha_1(v,[[\xi_1,v],v])]$

The expression $-d\alpha(v,[\xi,v])$ is exactly $\frac{\beta\wedge d\beta}{\alpha\wedge d\alpha}$. The computation in the Lemma above shows that the positivity of this expression is preserved through convex-combination. That is, if $\frac{\beta\wedge d\beta}{\alpha\wedge d\alpha}$, with $\beta = d\alpha(v,.)$, is positive for $\alpha = \lambda\alpha_0$ and for $\tilde{\alpha} = \mu\alpha_0$, λ,μ positive functions, then given $t\in[0,1]$, it is also positive for $\alpha_t = \frac{1}{\frac{1-t}{\lambda}+\frac{t}{\mu}}\alpha_0 = \lambda_t\alpha_0$.

We cannot use this Lemma as such, with $\lambda=\lambda$ and $\mu = 1$ because $\lambda\alpha_0$ is convex (with respect to v) only on a small neighborhood of the boundary of one of the two tori and so is α_0 on a small neighborhood of the other torus. Therefore the function t has to depend on $1-x^2-y^2$, falling to zero as x^2+y^2 tends to 1 and being essentially equal to 1 or 0 elsewhere.

We then find by Lemma 1 above-$\lambda_t = \frac{1}{\frac{1-t}{\lambda}+t}$- after rescaling v so that $d\alpha_0(v,[\xi_0,v] = -1$:

$$-d\alpha_t(v,[\xi_t,v]) = \lambda_t((1-t)(\frac{1}{\lambda}+(\frac{1}{\lambda})_{vv}+(\frac{1}{\lambda})_v d\alpha_0(v,[[\xi_0,v],v]))+t+t_{vv}-\frac{t_{vv}}{\lambda}-2t_v(\frac{1}{\lambda})_v)+(t_v-\frac{t_v}{\lambda})d\alpha_0(v,[[\xi_0,v],v]))$$

We can choose t to be a function ω of u, $u = 1-x^2-y^2$, with $\omega(0) = 0, \omega(\epsilon) = 1$, $\omega\in L^\infty$, valued in $[0,1]$, $\omega'(u)u$ and $\omega"(u)u^2 = o(1)$.

Observe that, with $v = f(x)\times$our former v, $f\in C^\infty$, we have:

$$t_v = -2f(x)\omega'(u)(xdx+ydy)(v) = 2f(x)\omega'(u)yu = o(1)$$

and

$$t_{vv} = o(1)+f(x)(-2\omega"(u)y^2u^2+2\omega'(u)dy(v)u-2\omega'(u)y^2u = o(1)$$

It follows that $-d\alpha_t(v,[\xi_t,v])$ is also positive if it is positive for α_0 and $\lambda\alpha_0$.

We have completed the gluing up of the two forms into a "convex" form α_t (relative to v), satisfying $d\alpha_t(v,[\xi_t,v]) \lneqq 0$, so that if $\beta_t = d\alpha_t(v,.)$, then $\frac{\beta_t\wedge d\beta_t}{\alpha_t\wedge d\alpha_t} \gneqq 0$

α_t is of course a contact form corresponding to the standard contact structure on S^3. We will denote it α in the sequel and we will denote β_t β.

Fact is that even though $\beta\wedge d\beta \gneqq 0$, v is in the kernel of a non-singular foliation γ transverse to $ker\alpha$. This simple geometric fact implies strong compactness properties.

Indeed, we have the following technical result:

Lemma 2. *Let γ_s be the one parameter group of v. Let x_0 be a point of M and let x_s be $\gamma_s(x_0)$. Then*

$$\alpha_{x_s}(D\gamma_s(\xi(x_0))) \lneqq 1$$

Proof. Assume that there exists e.g s_0 positive such that

$$\alpha_{x_{s_0}}(D\gamma_{s_0}(\xi(x_0))) = 1$$

Observe that $\alpha_{x_0}(\xi(x_0)) = 1$. Setting
$\alpha_{x_s}(D\gamma_s(\xi(x_0))) = \phi(s)$, we have: $\phi'(s) = d\alpha_{x_s}(v, D\gamma_s(\xi(x_0))) = \beta(D\gamma_s(\xi(x_0)))$ and $\phi"(s) = d\beta(v, D\gamma_s(\xi(x_0))$ so that $\phi"(0) = d\beta(v, \xi(x_0)) = -d\alpha(v, [v, \xi(x_0)]) \lneq 0$

Thus $\phi(0) = 1$ is a local maximum and $\phi(s)$ is a decreasing function at first until it reaches a minimum. At this minimum $\bar{s}$ ($\bar{s} \lneq s_0$), ϕ' is zero, $D\gamma_{\bar{s}}(\xi(x_0))$ is, up to a v-component, collinear to $\pm\xi(\gamma_{\bar{s}}(x_0))$. Since $\phi"(\bar{s}) \gneq 0$, $D\gamma_{\bar{s}}(\xi(x_0))$ must be positively collinear to $-\xi(\gamma_{\bar{s}}(x_0))$ and $\phi(\bar{s})$ is negative.

Thus, $\phi(s)$ must have two zeros in the interval $(0, s_0)$, one s_1 in $(0, \bar{s})$, the other one s_2 in $(\bar{s}, s_0)$.

At each of these zeros, $D\gamma_s(\xi(x_0)$ is collinear to $[v, \xi](x_s)$. Since $D\gamma_s(\xi(x_0)$ is a v-transported vector, $[v, \xi](x_{s_1})$ is mapped onto $\lambda[v, \xi](x_{s_2}) + \mu v$ through $\gamma_{s_2 - s_1}$ and $ker\alpha_{x_{s_1}}$ is mapped onto $ker\alpha_{x_{s_2}}$.

This is impossible because this would force $ker\alpha$ to coincide with the kernel of the foliation γ (which is v-transported) somewhere between s_1 and s_2. Lemma 2 follows. Some more thought shows that in fact $1 - \alpha(D\gamma_s(\xi(x_0)))$ must be bounded away from zero.

In view of the criticality conditions that have to be verified by a critical curve at infinity, [2], Proposition 41, pp 251-252, [3], pp 107-109 , we conclude from Lemma 2 and its proof that, if a curve is critical at infinity, then all its ξ-pieces must be characteristic. Assuming that one of the $\pm v$-jumps of this critical point at infinity is infinite, it must have at least one of its "edges" (these might not be well defined) near one of either O_1 or O_2. We prove below, see subsection 1 of section 2, that the piece of curve after this $\pm v$-jump until the next infinite $\pm v$-jump depends at most on one parameter. This next infinite $\pm v$-jump also "starts" or "ends" "on a periodic orbit" (brackets are introduced to emphasize the fact that these ends might not be well defined). Let us assume that the first very long/infinite $\pm v$-jump does not start **and** end on O_1 or O_2. Assume for example -without loss of generality- that it ends but does not start at O_1 or O_2. We use then Lemma 3 below to exclude the case where the intermediate history is reduced to a characteristic piece. Thus, there is an intermediate finite $\pm v$-jump. The arguments of Lemma 3 can be extended to rule out this occurrence[2]. This implies that all very long/infinite infinite $\pm v$-jumps start and end "on periodic orbits". The history between two such very long/infinite $\pm v$-jumps cannot be reduced then to a single characteristic piece. Starting from the first periodic orbit related to the first of these two infinite $\pm v$-jumps and going to the other (maybe the same) periodic orbit corresponding to the second infinite $\pm v$-jump, we have at most three free parameters: each of the periodic orbits provides a free parameter, since we do not know where the edge is precisely located, and there might also be the free parameter of the intermediate history. Setting up the intersection problem across the finite v-jump (it has a "limit" as the intermediate free parameter of the history tends to infinity, observe that we may assume for a Palais-Smale sequence that the total length along ξ is bounded and observe that v is Morse-Smale), we derive by transversality that the edges are in fact precise, to be chosen among a finite number of points, as well as all intermediate edges. Then, the $\pm v$-jumps cannot tend to infinity over a continuous process. Theorem 1 follows.

2. Morse-Smale vector fields of $ker\alpha$ (Proof of Theorem 2).

We now consider the case of a Morse-Smale vector-field v in $ker\alpha$ which is non-singular and has, for simplicity, just one attractive periodic orbit O_1, one repulsive periodic orbit O_2 and a number of hyperbolic periodic orbits. Among these, we may distinguish between those along which $ker\alpha$ "turns well" (see [1] p I.11) along v and those along which $ker\alpha$ does not "turn well". $ker\alpha$ "turns well" along v if, starting from any point x_0 of the periodic orbit, the trace of $ker\alpha$ completes, in a v-transported frame, at least one (hence infinitely many) revolution. For the latter, that is if $ker\alpha$ does not "turn well" along v, there is a unique model, see [9], Appendix 3.

[2]Assume, for simplicity, that this intermediate finite $\pm v$-jump is directed along $+v$. Using exactly the same argument than in Lemma 3 below, we find that ξ, $-v$-transported from its base to its top must be collinear to ξ up to $o(1)$. Thus this intermediate history is rigid. This implies, using a dimension argument, that the starting point of the second infinite $\pm v$-jump is not on a periodic orbit. We then transport ξ along $+v$ from the base of this intermediate v-jump to its top. We $\pm v$-transport ξ, as in Lemma 3, from the edge "on a periodic orbit" to the other edge (not on a periodic orbit) along the second infinite $\pm v$-jump and we match (if possible). Arguing as in Lemma 3, either this second very long/infinite $\pm v$-jump is finite or the intermediate $\pm v$-jump is subject to a second independent condition, we rule this out by genericity, or the first infinite $\pm v$ jump is finite because its end-point is precise while its initial point is "far" from O_1 and O_2.

If there are no such hyperbolic periodic orbits, then, using the differential equation (1)-see [9], Lemma 1 and Corollary 1, pp44-45 to understand how to use (1), see also section 5 of this paper, v might be slightly modified near O_1 and O_2 for the proper use of (1)-:

$$[v,[v,\xi]] = -\xi + \gamma(s)[\xi,v] - \gamma'(s)ds(\xi)v$$

where s is the time along v and using the fact that $ker\alpha$ turns well everywhere, we can derive a form $\lambda\alpha$ in our given contact structure such that $\beta = d(\lambda\alpha)(v,.)$ is a contact form with the same orientation than α.

We will see-it is a byproduct of our more elaborate proofs below-that the Palais-Smale condition is satisfied then for all flow-lines originating at the periodic orbits of ξ in the spaces Γ_{2k} of closed curves made of $k\xi$-pieces of orbits alternating with $k \pm v$-pieces of orbits.

In order to attain greater generality, we need to address the possibility that there may be hyperbolic periodic orbits for v around which α does not "turn well".

This is what we will consider now; and in order to make the insight deeper and the proofs lighter, we will consider the case when v has three periodic orbits, one attractive, one repulsive and a third hyperbolic one, around which α does not "turn well"- see section 4 for simple observations leading to the general case.

Rescaling the rotation of β, that is redefining α along v-orbits with the help of the differential equation (1), we may assume that β has the same orientation than α, except on a small hyperbolic neighborhood V of the hyperbolic orbit O_3. In this neighborhood, β has the opposite orientation and on ∂V, $\beta \wedge d\beta$ is zero. We will see later how to build V more specifically. These specifications are not needed for what immediately follows.

1. History, Criticality conditions.

Let us first analyze the behavior of a critical point at infinity and track its history, assuming that this critical point at infinity has a very long/infinite $\pm v$-jump. We disregard for the moment the fact that this $\pm v$-jump could be made of different very long/infinite pieces of $\pm v$-orbits as could happen along a process where a tiny ξ-piece separates two very long $\pm v$-jumps over nearly critical curves, see again section 4 for the more general case.

The history of a critical point at infinity, of a critical point of the functional J_∞ on the spaces Γ_{2k}s can be expected to be completely "deterministic". To have an idea of what this history is, we will consider instead the functional $I_\infty(x) = \Sigma a_i$. I_∞ is J_∞ when $\beta \wedge d\beta$ is positive but can be defined even when $\beta \wedge d\beta$ is not everywhere positive, as in V here. The transition from I_∞ to J_∞ is explained in section 4.

A critical point of I_∞ on a Γ_{2k} is completely determined by one of its points as long as its $\pm v$-jumps are finite. Indeed, whenever we consider a $\pm v$-jump of a critical point at infinity, we also consider the two ξ-pieces abutting a this $\pm v$-jump. If these two ξ-pieces are non-degenerate, then the form α should be transported onto itself through v-transport from one edge to the other edge of the $\pm v$-jump. This imposes a strong restriction on these end points which have then to belong to "characteristic" hypersurfaces in M.

If only one ξ-piece is non-degenerate and the other one is "characteristic", then the $\pm v$-jump is subject to the condition that the image z of ξ after v-transport along the $\pm v$-jump from the characteristic to the free ξ-piece should satisfy $\alpha_x(z) = 1$. In addition, there is another criticality condition that involves this $\pm v$-jump and the characteristic piece, see [3], pp 107-109[3]. We describe for further use this condition: Assuming that the other $\pm v$-jump abutting to the characteristic piece is free, its other end point is determined by the condition described above, on the other side: that ξ, v-transported from the characteristic piece to the free ξ-piece, is mapped onto z' with $\alpha_x(z') = 1$. We then have z on one side and z' on the other side. We can scale z into cz so that this vector, ξ-transported across the characteristic ξ-piece, coincides with z' up to some ξ-component. We derive in this way a tangent vector and the first variation of I_∞ on it should be zero. Writing $z = A_1\xi + B_1w + C_1v$ and $z' = A'_1\xi + B'_1w + C'_1v$, w in $ker\alpha$ and writing that v is mapped onto θv from one end to the other of the characteristic ξ-piece in the ξ-transport, we find that:

$$A_1B'_1\theta = A'_1B_1$$

[3]these criticality conditions have been derived in [2], Proposition 41 and [3], pp 107-109, under the assumption that $\beta \wedge d\beta \gtrsim 0$. However, for I_∞, they readily extend even when this condition is violated. We will be using them as such in this paper.

Since the other end of the other $\pm v$-jump abutting to the characteristic ξ-piece is determined by the equation $\alpha_x(z') = 1$, we may view this condition as a condition on the other end of the first $\pm v$-jump, which is therefore completely determined.

It follows that, starting from a ξ-piece, we can follow this history, and in the case of a critical point at infinity which has infinite $\pm v$-jumps, starting from this ξ-piece until the next infinite $\pm v$-jump, the curve is completely determined up to one single free parameter. This parameter has to be introduced because we cannot, at this point, make sense of any criticality condition across the infinite $\pm v$-jump: either the "last" ξ-piece abutting to the infinite $\pm v$-jump is free and its end point is not subject, at this point, to any condition. The length of this non-degenerate ξ-piece is then a free parameter. Or this "last" ξ-piece is characteristic. Then the $\pm v$-jump preceding it is not subject to a condition across the characteristic piece at this point; that frees one parameter, unless the preceding ξ-piece is free; then this $\pm v$-jump is determined, but we have one free parameter(the length of this ξ-piece)etc.

2. Infinite $\pm v$-jumps not crossing V.

In our framework, the starting and the ending point of an infinite $\pm v$-jump must be either in $W_u(O_3)$, the unstable manifold of the hyperbolic orbit O_3 or in $W_s(O_3)$, the stable manifold of O_3, or it must be a point on a v-periodic orbit. Indeed:

Lemma 3. *Let x^∞ be a critical point at infinity. Then x^∞ cannot contain two infinite $\pm v$-jumps connecting points of M-$O_1 \cup O_2$ to periodic orbits, not intersecting the neighborhood V of the hyperbolic orbit O_3 and separated by a characteristic piece*

Proof. Considering ξ at the other end of the infinite $\pm v$-jump than the periodic orbit, we v-transport it to the end on the periodic orbit, which is attractive or repulsive and ξ collapses thus into a very small vector z. Considering now the other $\pm v$-jump, assuming it is infinite, we v-transport ξ from the end lying on the periodic orbit to the end on the characteristic piece. This becomes a large vector which we split into $A_1\xi + B_1 w + Cv$, w in $ker\alpha$. We claim that $\frac{B_1}{A_1}$ needs to be small, otherwise we could build a tangent vector after ξ-transporting this vector to the other hand of the characteristic piece, scaling it so small that its w component equals that of z. The ξ and v-components are easy to adjust then. The variation of I_∞ on this vector is far from zero because the scaling of B_1 into a small value implies that A_1 scales then also into a small constant and so does the vector ξ, into $\epsilon\xi$, taken at the end of the second $\pm v$-jump. x^∞ cannot be critical. Observe that this argument repeats if the second $\pm v$-jump is finite and B_1 is non-zero. We'll use this fact later.

Now, $\frac{B_1}{A_1}$ being very small implies that ξ is v-transported onto a vector almost parallel to $\pm\xi$ between the endpoints of the $\pm v$-jump. Assume now that the second end of this $\pm v$-jump is on a periodic orbit around which $ker\alpha$ "turns well". Then, over a continuous process, as the curve comes close to x^∞ and its $\pm v$-jumps become larger and larger, the restriction that ξ is v-transported onto a vector almost parallel to $\pm\xi$ translates into a bound on this $\pm v$-jump, which cannot therefore become infinite (the rotation of ξ has to remain close to an integer, therefore bounded on a continuous process). x^∞ cannot therefore be the end of a flow-line for example and is not therefore a critical point at infinity.

If, on the other hand, $ker\alpha$ does not "turn well" around this periodic orbit, then we may assume that the Poincare-return map of v along this periodic orbit has two invariant foliations, u_1 and u_2, both corresponding to negative eigenvalues $\lambda_1 \lneq \lambda_2$ and we may assume that ξ is transverse to both foliations along the periodic orbit. Then, since ξ has to be mapped onto a vector collinear to ξ up to$o(1)$ from one end to the other of the second $\pm v$-jump, this forces ξ at the initial point to be collinear to u_1 up to $o(1)$. This initial point, we know by transversality not to be close to a periodic orbit. Thus it must live on a compact hypersurface of M. It is also the end-point of a characteristic ξ-piece starting at a periodic orbit. Setting up the intersection problem, we find that the end-point of the first very long/infinite $\pm v$-jump, and the starting point of the second very long/infinite $\pm v$-jump are "precise", to be chosen among a finite number of choices. The first very long/infinite $\pm v$-jump cannot then be infinite since it is not starting at a periodic orbit.

Another result is in fact much more obvious than this one. It reads:

Lemma 4. *x^∞ cannot contain an infinite $\pm v$-jump connecting a point of M to a periodic orbit of v outside of V followed by a non-degenerate ξ-piece.*

Proof. ξ, v-transported from one end of the $\pm v$-jump to the end on the periodic orbit, collapses into a tiny vector z transversally to v and z, being tiny, cannot satisfy the equation $\alpha(z) = 1$.

In the next several sections, we enter into the proof of Theorem 2 as we consider more complex potential critical curves at infinity which might involve very long/infinite $\pm v$-jumps crossing V.

3. All very long/infinite $\pm v$-jumps are "not on the v-periodic orbits".

Let us consider critical points at infinity with very long/infinite $\pm v$-jumps. If one of these very long/infinite $\pm v$-jumps has points of the attractive or the repulsive periodic orbits in its closure as a topological set,we will say that this $\pm v$-jump has edges "lying on a v-periodic orbit". Critical points at infinity having all the edges of all their very long/infinite $\pm v$-jumps not"on periodic orbits of v" are the simplest to rule out. And that is what we will be doing in a first step.

We will first consider such critical points at infinity under the additional condition that all their ξ-pieces are non-degenerate. We then, through a sequence of simple observations, will generalize this result to include critical points at infinity that support, in between any two consecutive very long/infinite $\pm v$-jumps, a history which includes a free parameter. We will make this statement clear in time.

Next, we will rule out such critical points at infinity allowing for some definite, determined histories between a pair of consecutive very long/infinite $\pm v$-jumps, but still not allowing for the edges of such $\pm v$-jumps to lie "on v-periodic orbits". Typically, such a definite history is given by a characteristic ξ-piece connecting two such infinite $\pm v$-jumps.

In the last subsection (subsection 4), we will allow for the edges of such infinite $\pm v$-jumps to lie on "v-periodic orbits". The proofs, though rigorous are only sketched as far as transversality issues are involved. Full details can be carried out by the conscientious reader and will also be published elsewhere [10].

1. All ξ-pieces are non degenerate

When we assume that the edges of the infinite $\pm v$-jumps of a curve of Γ_{2k} are "not on a v-periodic orbit" and that all of its ξ-pieces are non-degenerate, the proof of the fact that it cannot be a critical point at infinity is very clear.

We use the expression "not on a v-periodic orbit" with great precaution because we are considering infinite $\pm v$-jumps and these might not have definite edges. We also rule out for the time being infinite $\pm v$-jumps that might be made up with several infinite $\pm v$-pieces; this might happen typically with two very long $\pm v$-jumps separated by a tiny ξ-piece, the tiny ξ-piece vanishing over the process as the two very long $\pm v$-jumps become infinite. We will exclude these occurrences later (section 4).

A very long /infinite $\pm v$-jump whose edges are not "on a v-periodic orbit" must be in $(W_u(O_3) \cup W_s(O_3)) - (T_1 \cup T_2)$. T_1 and T_2 are the two (small) solid tori around the attracting and the repulsive periodic orbits of v.

We claim now that:

Lemma 5. *Assume that all the ξ-pieces are free and all the $\pm v$-jumps are "not on periodic orbits". Then this curve cannot be the end of a decreasing flow-line for a pseudo-gradient of I_∞.*

Proof. Let us consider a very long/infinite $\pm v$-jump and let us assume in a very first step that the ξ-pieces starting and abutting at this very long/infinite $\pm v$-jump are free. Then these edges are subject [2] to two criticality conditions. These two conditions read, starting from the edge $x^\pm$, with $s^\pm$ the time along v:

$$\alpha_{x^\pm}(d\gamma_{s^\pm}(\xi(x^\pm))) = 1$$

$x^\pm$ are, each, in $W_u(O_3) \cup W_s(O_3)$ which is a space of dimension 2.

These two conditions are effective and independent, see Proposition 1 , below. (x^+, x^-) varies therefore in a space of dimension 4, subject to two independent conditions. Enforcing them, we find that (x^+, x^-) lives in a two dimensional stratified space.

In fact, each of x^+ and x^- lives on a one dimensional stratified subset of $W_u(O_3) \cup W_s(O_3)$. This follows from Proposition 1, below .

We take x^+ or x^-, indifferently, and we evolve through the attached ξ-piece to the other vertex and so forth, until we reach again $W_u(O_3) \cup W_s(O_3)$. The history of the curve in between is completely determined, up to a finite number of choices and one single free parameter, which can either be the length of the first or the last ξ-piece in between or, if these are characteristic ξ-pieces, the length of the first or last finite $\pm v$-jump in between.

The constraints on one vertex transform in this way in as many constraints on the other vertex, that is on restrictions on the dimension of the stratified subset of $W_u(O_3) \cup W_s(O_3)$ where this vertex can live.

We now have a new infinite $\pm v$-jump in $W_u(O_3) \cup W_s(O_3)$ and there are two additional constraints on its edges, which are independent from the constraints derived by transport from the previous very long/infinite $\pm v$-jump. These constraints are "split", that is they are read as constraints on one of the vertices at a time (see Proposition 1 for these two last constraints, for those transported, the claim is obvious).

One of the edges is therefore one of a finite collection of points, while the other one lives in a stratified subset of $W_u(O_3) \cup W_s(O_3)$ of top dimension 1.

Proceeding in this way, step by step, we derive, after "closing the loop" around the very long/infinite $\pm v$-jumps of the curve, that each edge is in fact a fixed point, to be chosen in a finite set of points because there is at least one constraint on the other edge of each very long/infinite $\pm v$-jump in $W_u(O_3) \cup W_s(O_3)$, that is on the edge which is not facing the edge of the previous $\pm v$-jump of the same nature through the determined history that we described earlier.

Of course, the argument is greatly simplified by the fact that we are assuming that these are the only very long/infinite $\pm v$-jumps.

Also, the constraints on each separate very long/infinite $\pm v$-jump are "split": they read " w at $x^{\pm}$ is parallel to $\gamma^{\pm}$" (without correspondence between the $\pm$s), or they read "ξ is parallel to $\gamma^{\pm}$". If two such constraints are imposed on the same edge, one will read for example "w is parallel is $\gamma^{\pm}$" as the other one will read "ξ is parallel to the same γ^{+}" (see Proposition 1). Such constraints have no intersection.

To conclude, we just observe that, over a continuous process, if x^+ and x^- are constrained to be among a finite number of choices, then the related $\pm v$-jump cannot be infinite and the Palais-Smale condition holds under this special form (over a continuous process).

This concludes the proof under the assumption that all ξ-pieces are free and all infinite $\pm v$-jumps are "not on v-periodic orbits".

This result generalizes as follows:

Lemma 6. *Lemma 5 holds if the assumption that all the ξ-pieces are free is replaced by the assumption that the history of each piece of the curve between any two consecutive very long/infinite $\pm v$-jumps (which we assume "not to be on periodic orbits") is "free".*

Proof.

When the history of the curve between two infinite $\pm v$-jumps in $W_u(O_3) \cup W_s(O_3)$ contains characteristic pieces, in particular when these characteristic ξ-pieces are attached to these infinite $\pm v$-jumps, this argument generalizes as long as "one parameter is free", that is as long as this history does not obey a finite, rigid family of patterns, independent of the initial and final points in $W_u(O_3) \cup W_s(O_3)$.

Indeed, we may assume then that this history contains finite $\pm v$-jumps, typically we would have an infinite $\pm v$-jump of $W_u(O_3) \cup W_s(O_3)$, a characteristic piece, then a finite $\pm v$-jump to which another characteristic piece, abutting in $W_u(O_3) \cup W_s(O_3)$ again, would be attached.

We take ξ from this ξ-piece, from the edge where the finite $\pm v$-jump abuts; we v-transport it back to the "top" of the finite $\pm v$-jump and then ξ-transport back it to the edge of the very long/infinite $\pm v$-jump. We derive a vector $z = \delta\xi + \mu w + \nu v$. If μ is non zero, then the characteristic piece followed by the finite $\pm v$-jump acts as "free" ξ-pieces were acting in our previous arguments, only that as we glue up z properly scaled with a v-transported vector along the very long/infinite $\pm v$-jump, typically $\gamma^{\pm}$, see Proposition 1 below, we will derive, instead of a condition such as "$\gamma^{\pm}$ is parallel to w", another condition which takes into account the fact that the construction of z involve a certain amount of variation along ξ.

Namely, z, when ξ-transported on the other side of the characteristic ξ-piece, reads as $z_1 = (1 + A_1)\xi + B_1 w + C_1 v$.Let θ be the component along v of the vector v after being ξ-transported along the characteristic ξ-piece. We then find when we express criticality instead of the condition "$\gamma^\pm$ is parallel to w" (this condition appears below, in the proof of Proposition 1), the condition:

$$\gamma^\pm \in ker(\alpha - \frac{A_1\theta}{B_1}\beta)$$

Starting then from $W_u(O_3) \cup W_s(O_3)$, at $x^\pm$ where $\gamma^\pm$ is given, we find that the above condition subjects the finite $\pm v$-jump to a constraint. The history of the curve between the two consecutive very long/infinite $\pm v$-jumps becomes rigid, determined up to a finite number of choices, independent of the end point.

But this endpoint must be on $W_u(O_3) \cup W_s(O_3)$ and the related intersection problem imposes one constraint on $x^\pm$ and on the other vertex, at the end of the partial history. The two end points are constrained to live on a stratified subset of $W_u(O_3) \cup W_s(O_3)$ of top dimension 1 and the compactness argument therefore generalizes under this new assumption.

Our proof is now complete under the more general assumption that the "history" between two very long/infinite $\pm v$-jumps is "free" and that these jumps do not have their edge points on "v-periodic orbits".

2. Some intermediate histories are not "free".

The proof of the compactness result in this more general framework (we are still assuming that our $\pm v$-jumps are in $W_u(O_3) \cup W_s(O_3)$) requires Proposition 1, below.

To illustrate this Proposition, we will first consider the simple case of a very long/infinite $\pm v$-jump of $W_u(O_3) \cup W_s(O_3)$ followed by a characteristic ξ-piece, followed again by a very long/infinite $\pm v$-jump of $W_u(O_3) \cup W_s(O_3)$. We assume that this piece of curve is preceded and followed by non-degenerate ξ-pieces. We then observe that, on this piece of curve, three criticality conditions hold. Namely, on each side, starting from the characteristic ξ-piece and going to the bottom of the $\pm v$-jump, ξ should be $\pm v$-transported into a vector z satisfying $\alpha(z) = 1$. This provides two conditions. A third condition states that $\partial I_\infty.z_1 = 0$; this third tangent vector z_1 is derived using this characteristic piece by transporting ξ **backwards** from the other end of the $\pm v$-jumps to the characteristic piece and, then matching the two transported vectors, after scaling, over this characteristic ξ-piece (the v-components are easily adjusted using the edge $\pm v$-jumps, the ξ-components are adjusted by ξ-transport, the w-components need to be matched).

We write each of these conditions using the $\gamma^\pm$-hyperbolic foliations of O_3. Using these foliations, we claim that each of these conditions "localizes", providing one constraint on the "bottom" edge of the first $\pm v$-jump and one constraint on the "top"edge of the last one, while the third condition provides a constraint on the two remaining edges. This result is in fact more general and we claim that:

Proposition 1. *Given a very long/infinite $\pm v$-jump in $W_u(O_3) \cup W_s(O_3)$,*

i)if this $\pm v$-jump is preceded and followed by non-degenerate ξ-pieces or more generally, if the histories preceding and following this $\pm v$-jump are "free", then each edge of this $\pm v$ is subject to one condition that constrains it to live on a stratified subspace of $W_u(O_3) \cup W_s(O_3)$ of dimension 1.

ii) if this $\pm v$-jump is part of a maximal sequence of very long/infinite $\pm v$-jumps, all of them in $W_u(O_3) \cup W_s(O_3)$ and separated by constrained intermediate histories, with the first(respectively the last) $\pm v$-jump of the sequence being preceded (respectively followed) by a "free" history, then orienting these histories according to the curve, the bottom edge of the first $\pm v$-jump and the top edge of the last $\pm v$-jump in the sequence are forced to live on a stratified subspace of $W_u(O_3) \cup W_s(O_3)$ of dimension 1, while all the other intermediate edges are precise, to be chosen among a finite number of choices.

Proof.

We start with the case described above of a single characteristic piece separating two very long/infinite $\pm v$-jumps of $W_u(O_3) \cup W_s(O_3)$. Let us consider the two first conditions. They are of the same nature: we take ξ at one end of a very long $\pm v$-jump, the end that abuts to the characteristic piece. This ξ reads, using the (γ^+, γ^-, v)-frame as $A\gamma^+ + B\gamma^- + Cv$. Assume, for the sake of simplicity, that γ^+ is the expanding direction and let s_3 be the length of the very long/infinite $\pm v$-jump. Then, the transport of ξ along $\pm v$ to the other end of this $\pm v$-jump reads $z = Ae^{s_3}\gamma^+ + Be^{-s_3}\gamma^- + Cv$. Since $\alpha(z)$ should be 1, we find that $A\alpha(\gamma^+) = O(e^{-s_3})$. It follows that either A is very small or $\alpha(\gamma^+)$ is very small. These conditions can be read as conditions on one edge of the very long/infinite $\pm v$-jump. They neither involve the entire $\pm v$-curve nor the remainder of the piece of curve. The first condition is a condition on the end point of the $\pm v$-jump where the characteristic piece abuts, the other one is on the other end of the $\pm v$-jump, there where the non-degenerate ξ-piece abuts.

We move now to the third condition, which we would like to read as a condition on the end points of the characteristic piece. Reading ξ at each end of either $\pm v$-jumps not abutting to the characteristic piece as $A_1\gamma^+ + B_1\gamma^- + C_1 v$ (with different values of A_1, B_1, C_1 on each side), we $\pm v$-transport it up to the ends of the characteristic piece. Now, again for the sake of simplicity, γ^- is the expanding direction. So that our vectors, once transported back, read essentially as $B_1\gamma^- + o(1)$, up to some v-component which we can ignore since we can adjust those using the $\pm v$-jumps and $\alpha(v)$ is zero; this if B_1 is not zero on each side.

There is now an $e^{s_3}B_1\gamma^-$ on each side. If one of them has a w-component which is not o(1), we can use to adjust these two edge data into a large vector z_1. If $\alpha(\gamma^-)$ is not o(1) at both ends, the criticality condition reads on these values $e^{s_3}\alpha(\gamma^-)$, the initial ξ that we used in the process contribute O(1) in the computation of the variation $\partial I_\infty.z$ of I_∞ along this direction z and therefore, since each e^{s_3} is very large, the condition reads as a condition on the edges of the characteristic piece.

Observe that the condition that "$\alpha(\gamma^-)$ is not o(1)" at the edges of the characteristic piece is also a condition localized at these edges.

Observe that if, at the previous step, as we are enforcing the two first conditions, one of these conditions reads as $\alpha(\gamma^+) = o(1)$, enforced at the edge of one of the $\pm v$-jumps where the non-degenerate ξ-piece abuts, then the direction γ^+ is essentially in $ker\alpha$; consequently B_1 cannot be zero at this point and the third condition, on this side, does not translate into another condition on the same edge.

Thus, our conditions never translate into two conditions on a "bottom edge". On the other hand, the top edges, being edges of a characteristic piece or of a piece of curve having a precise history from $W_u(O_3) \cup W_s(O_3)$ to itself, are forced to vary into a stratified space of top dimension 1 and consequently not more than one condition can be imposed on them.

It follows that our criticality conditions lead us to one possible configuration: one condition on each "bottom edge", forcing this edge to vary in one dimensional stratified subsets of $W_u(O_3) \cup W_s(O_3)$; the last condition then reads on the "top edges", which are then precise, to be chosen in small neighborhoods of a finite number of points.

When there are more than one characteristic piece between the two non-degenerate ξ-pieces, we claim that the result generalizes and the edges, all over the sequence, are precise, to be chosen among a finite number of choices, except for the first and last edge, which are constrained to live on a stratified subspace of top dimension 1.

Indeed, when there are more than one characteristic ξ-piece in the sequence, for each sub-piece made of a characteristic ξ-piece in the sequence and its adjacent very long/infinite $\pm v$-jumps, we do not have three conditions, but only one, except for the two edge such sub-pieces, upon which two conditions are imposed.

These conditions "localize" as above, translating into conditions on edges. We can never have two conditions on a single edge, because either these edges are internal edges; they are also endpoints of a characteristic ξ-piece between $W_u(O_3) \cup W_s(O_3)$ and itself and they are constrained to live on a stratified subspace of top dimension 1. Or there are end-edges and our arguments above apply to them.

We enforce our conditions on the two first sub-pieces. These are three conditions. Suppose that they translate into the fact that the first edge is constrained to live in a stratified space of top dimension 1 while the fourth, fifth (automatically, it is the other end of a characteristic piece that has its first end to be the fourth edge) and sixth edges (hence seventh as well, if it exists) are precise. Then all the other edges in the sequence are precise, because the only choice is to have the three conditions enforced on the first edge (constrained therefore to live in a stratified space of top dimension 1), on the fourth edge and on the sixth edge. Thus the next criticality condition on the sequence, the one on the third sub-piece cannot but provide a condition on the eight edge, since all the other edges of this third sub-piece are already precise. And so forth, all edges become precise, the last one because, even though it has one additional degree of freedom, it is also subject to an additional condition. This last point is thus then subject to two conditions and by our arguments above, we know this to be impossible.

The same argument holds if the imposition of these three conditions on the two first sub-pieces translate into the fact that the edges are all precise in these two sub-pieces except for the first one, which would not be subjected to any condition.

Thus, our first edge is subject to one condition, the second (and third) as well. Thus either the fourth and the fifth are not. By the same arguments as above, all the remaining edges are precise. Or they are, and the outcome of this analysis on the two first sub-pieces is that the the sixth edge is not determined by these three conditions. If another sub-piece is involved, an induction starts. The conclusion is that either the endpoints of one characteristic piece are not precise through this (partial) enforcement of criticality conditions process; then all the other edges after this characteristic piece are precise, ending edge included. This yields a contradiction because this ending edge is then subject to two conditions.

The conclusion is that all the edges, over the sequence, are precise, to be chosen among a finite number of choices, except for the first and last edge, which are constrained to live on a stratified subspace of top dimension 1. This proves ii).

Observe that if, on the other hand, at the two edges of a very long/infinite $\pm v$-jump, the ξ-pieces abutting are non-degenerate (or the like, the history of the curve bordering them is not precise, among a finite number of choices up to o(1)), then each of this edges is subject to one condition, namely that whatever direction in $\gamma^{\pm}$ at this point is contracting in the direction of the $\pm v$-jump should be in $ker\alpha$ (or a condition of the same type in the case of a precise history, in lieu of non-degenerate ξ-pieces). i) follows.

Proposition 1 immediately implies compactness, over continuous processes, when the attractive and repulsive periodic orbits are not involved in the definition of the critical point at infinity and when the associated curve has a non-characteristic ξ-piece. Indeed, by the argument above, all edges abutting to characteristic pieces (and the like) are precise, in a small neighborhood of a finite number of points. The other edges are subject to one condition. But such edges face each other in pairs; and the history of the curve in between gives rise to one condition on each edge. The condition on one of them translates by transport along the history into an additional, independent condition on the other one. The result is that each of these edges is, by transversality, constrained to be precise, in a small neighborhood of a finite number of choices. Accordingly, since the edges of each very long/infinite $\pm v$-jump are precise, this very long /infinite $\pm v$-jump cannot in fact be infinite. Here we are using strongly the fact that these $\pm v$-jumps are not abutting or starting at the attractive/repulsive periodic orbits. With these, the fact that the edges are precise does not automatically translate into the fact that the very long/infinite $\pm v$-jump is finite.

4. Some very long/infinite $\pm v$-jumps have one or both of their edges "on a v-periodic orbit".

In order to solve the remaining cases, which involve very long/infinite $\pm v$-jumps having points of the attractive or the repulsive periodic orbits in their closure, we have to study two special configurations:

First, we consider a very long/infinite $\pm v$-jump running from a point x_0 of $W_u(O_3) \cup W_s(O_3)$ (not close to the attractive or repulsive periodic orbits) to the attractive periodic orbit for example.

We introduce here an assumption, which we will later show to be always satisfied, after proper manipulation of v near the attractive and repulsive periodic orbits. Namely,

Assumption (A): we assume that, near those orbits, $ker\alpha$ "turns well along v", [1], pp I.11; that is starting from any point x_0 close enough to e.g the attractive periodic orbit of v, the kernel of α rotates at least once in the v-transport along the $+v$-orbit through x_0.

We will prove below (section 3) that, after possibly modifying v near these orbits, we may always assume that Assumption (A) holds.

We then claim, under such an assumption, that if such a $\pm v$-jump is preceded by a non-degenerate ξ-piece (or the like) in a piece of curve which is part of a critical point at infinity, then

Lemma 7. *a condition must be verified at x_0, forcing this point to live in a one dimensional stratified subset of $W_u(O_3) \cup W_s(O_3)$.*

Proof. This, for once, is not related to the criticality of this curve in its own Γ_{2k}, although we suspect that a proof for compactness could be built using only criticality conditions in Γ_{2k}. This condition is related to the fact that the curve is not a false critical point at infinity in the sense of [3], pp 111-112, Proposition 21, see also I.B.3 and I.B.4

pp 49-51 of [3] . It can rather be seen as a "criticality condition" in Γ_{2k+2} (or how the normal to Γ_{2k} in Γ_{2k+2} at our curve is I_∞-oriented).

Indeed, let us consider the portion of the very long/infinite $\pm v$-jump near the attractive orbit. Although the foliations $\gamma^\pm$ are not defined at the attractive and repulsive periodic orbits, we may assume that they are well defined, by v-transport, along the $\pm v$-jump since it lies in $W_u(O_3) \cup W_s(O_3)$. γ^- is the expanding direction as we travel away from the attractive orbit along $-v$. γ^- is v-transported, while $ker\alpha$, thus ξ (maybe with a few comebacks) rotates along v. Thus, $-\xi$ will coincide with $\pm\gamma^-$ an infinite number of times along the $\pm v$-jump. In fact, we can assume that, inside T_1 and outside a small , fixed solid torus T_1' around the attractive periodic orbit (observe that we are assuming that x_0 does not approach the periodic orbit, is not in T_1), coincidence of $-\xi$ with each of γ^- and $-\gamma^-$ will occur at least once in $T_1 - T_1'$.

Transporting back $\pm\gamma^-$ from such a point of coincidence to x_0 along $-v$, we find at x_0 a very large vector to x_0. Using the fact that the ξ-piece abutting to x_0 is non-degenerate, we build a "normal" to the curve pushing it into Γ_{2k+2}. This normal does not increase the number of zeros of b, only transforms the $*$ of the $\pm v$-jump into a family (see [5], we state this although it is irrelevant to our present argument because it could be useful in a more general perspective, when we use this result within our homology see [4],[5]). If it were I_∞-decreasing, the critical point at infinity would be false, we could bypass it at the expense of having part of our downwards variations being moved now to Γ_{2k+2}.

Since our vector at x_0 is so large, since this vector is read as $\pm Ce^{s_3}\gamma^-$, where the choice of $\pm$ is as we please, one of these normals will be I_∞-decreasing unless $\alpha(\gamma^-) = O(s_3)$ at x_0. This provides a condition on x_0.

In the last step in our proof of compactness, we establish the following result:

Lemma 8. *Assume that x_0, as above, is constrained to be in a small neighborhood of a given point in $W_u(O_3) \cup W_s(O_3)$. Assume that the ξ-piece following the $\pm v$-jump originating at x_0 is free (or the like). Then, this $\pm v$-jump cannot be infinite.*

Proof.

We can think of our $\pm v$-jump as being made of two pieces, a first piece from x_0 to ∂T_1, and a second piece inside T_1. The piece of v-orbit inside T_1 has to become infinite. The origin point in ∂T_1 of this v-orbit must remain in a small neighborhood of the trace of $W_u(O_3) \cup W_s(O_3)$ on ∂T_1, near a given connected component of this trace. Given a non-zero initial data in the tangent space to ∂T_1 in a small neighborhood of this trace, we can track the rotation of w with respect to the v-transported vector derived from this data along this second piece. We denote z this v-transported vector. The rotation of w with respect to z tends to infinity with the length of this second piece.

Over a continuous process, this can only happen if w and z coincide at one end or the other, or at both ends of this second v-piece infinitely many times.

The initial data for z will be the vector derived by v-transport of $\xi(x_0)$ from x_0 to ∂T_1. If z and w coincide at the end-point of the $\pm v$-jump, then denoting s the v-length of this v-jump,we find that :

$$\alpha(d\gamma_s(\xi(x_0))) = 0$$

at these occurrences and this violates one of the criticality conditions on the curve ([2], [3]).

Thus, this should not happen and the rotation of w with respect to z on this second piece should entirely be due to coincidences between w and z at its starting point, on ∂T_1. These coincidences are oriented coincidences, that is some of them could have the wrong orientation, but in total they should build a relative rotation tending to infinity.

Considering now the first piece of this v-jump over this continuous process, we can track the relative rotation of z and w over it. When O_3 is our special hyperbolic orbit, this relative rotation is bounded in function of the small neighborhood of the point in $W_u(O_3) \cup W_s(O_3)$ where x_0 is constrained to vary. $\xi(x_0)$ and w are given data at x_0, which are transverse. There are no coincidences at this end, coincidences building an infinite rotation at the other end on ∂T_1, thus a contradiction.

If on the other hand O_3 is another hyperbolic orbit, we can assume that w rotates monotonically in the same direction all over the v-jump and this can only occur over a continuous process if w and z coincide at the other end, yielding a violation of one of the criticality conditions, as above. Our claim is thereby established.

We now observe that we can remove the assumption that the ξ-piece following such a $\pm v$-jump is free or the like, because if it is not, then the history after this $\pm v$-jump until the next time the curve has an edge in $W_u(O_3) \cup W_s(O_3)$ is determined. Thus the end-point of the $\pm v$-jump is given. Over a continuous process along which this $\pm v$-jump becomes infinite, the rotation of w is then given, since the endpoints are given,or nearly so, so that no additional internal rotation can occur because of boundary coincidences. The $\pm v$-jump is, as above, divided into two pieces. There are two possible cases for O_3: either this orbit is the special hyperbolic orbit. Then, the amount of negative rotation of w around it over this first piece is finite, a priori bounded. Or O_3 is not this special hyperbolic orbit, we can assume that w then rotates around it positively (in the same direction than it does on the second piece).

In both cases, we conclude that the rotation along the second piece is finite and therefore that this $\pm v$-jump cannot have its other end "on the attractive or repulsive periodic orbit".

Compactness follows now each time a curve at infinity contains an infinite $\pm v$-jump connecting a point x_0 in $W_u(O_3) \cup W_s(O_3)$ to the attractive or repulsive periodic orbit. Indeed, we then claim that

Lemma 9. *the initial point x_0 has to be precise, to be chosen among a finite number of choices.*

This of course implies the conclusion.

Proof. In order to establish this fact, we consider the ξ-piece abutting at x_0, more generally the history of the curve before x_0, until the piece of curve has another edge on $W_u(O_3) \cup W_s(O_3)$ or one of the periodic orbits. Let us denote this other edge x_1. If x_1 is on one of the periodic orbits and this history is "free", then x_0 is on one hand constrained to be on the image of one of these orbits through this history, which forces it to vary in a one dimensional stratified subspace of $W_u(O_3) \cup W_s(O_3)$. On the other hand, using Lemma 8 above, one of $\gamma^{\pm}$ must be in $ker\alpha$ (or a similar condition) at x_0. The conclusion follows. Similarly, if the $\pm v$-jump abutting at x_1 connects it to one of the attractive or repulsive periodic orbits, then we can use Lemma 8 above. x_1 obeys a one dimensional constraint, which translates into a constraint on x_0, distinct from the constraint that it has to satisfy by direct application of Lemma 7. The conclusion follows also in this case, as well as in the case x_1 is on a periodic orbit and the history is precise, since then x_0, x_1 are precise, constrained to a finite number of choices.

We may thus assume that x_1 is on $W_u(O_3) \cup W_s(O_3)$ and that the $\pm v$-jump abutting at x_1 ends at x_2, again a point in $W_u(O_3) \cup W_s(O_3)$. Let us assume that the ξ-piece abutting at x_2 is non-degenerate (or the like). Let us assume also, without loss of generality, that γ^- is the expanding direction from x_2 to x_1. $\pm v$-transporting this direction, we derive a large vector at x_1 parallel to γ^-. If the history between x_1 and x_0 is "free", then this vector must be in the kernel of some form related to this history, see Lemma 6 above, typically the kernel of α if the ξ-piece starting at x_1 is non-degenerate. This provides a constraint on x_1. We also then have, because the history is "free", another independent constraint on x_0. Together, these two constraints force x_0 to be precise and compactness to hold.

We may thus assume that the history of the curve between x_1 and x_0 is precise, typically a characteristic ξ-piece. This forces x_0 and x_1 to obey a one dimensional constraint (holding for both together, they are linked in this constraint). At x_1, we have a very large vector along γ^-, but also at x_0, following the construction of Lemma 7 above, we have a very large vector. We can glue these two vectors, after transporting one of them to the other end of the history, following this history, unless their w-components are both zero; this is an occurrence that we can rule out, by genericity. We then express criticality using this global vector along this piece of curve (which actually pushes the curve in Γ_{2k+2}). Since it is a very large vector at x_0, x_1 and a bounded vector at the other "edges", criticality yields an additional condition at x_0, x_1, which are thereby precise and compactness follows.

The other possibility is that the history preceding x_2 is precise. This yields one constraint on x_2. Another one follows as we try to construct our vector as above. We pick up ξ at x_2. Either it obeys one constraint or we can build our vector, deriving a condition on x_0, x_1, hence compactness. If compactness does not hold, x_2 is precise, and so is therefore x_3. An induction starts, until we reach a "free" history again and we derive a contradiction. Compactness follows if the curve contains a $\pm v$-jump connecting a point of $W_u(O_3) \cup W_s(O_3)$ to the attractive or repulsive periodic orbit.

These arguments extend if a curve contains a $\pm v$-jump connecting the attractive periodic orbit of v and the repulsive one, when in addition the curve contains another history involving points of $W_u(O_3) \cup W_s(O_3)$ (it cannot involve $\pm v$-orbits connecting points of $W_u(O_3) \cup W_s(O_3)$ with periodic orbits of v by our previous argument). Tracking the intermediate ξ-pieces and arguing as above, we find that all edges are precise. Invoking the fact that the w-rotation along the $\pm v$-orbit connecting the attractive and the repulsive orbit is then bounded and the fact that this process is continuous, we conclude that this $\pm v$-jump cannot connect the attractive and the repulsive periodic orbits since $ker\alpha$ turns well near these orbits.

If the curve is made only of $\pm v$-jumps connecting periodic orbits of v and ξ-pieces in between, the edges are then forced to be precise and the fact that the w-rotation is then bounded over a continuous process implies the compactness again.

3. Assumption (A).

Let $D^2 \times S^1$ be the standard solid torus , $\{0\} \times S^1$ be its chore. Let $\alpha = yd\theta + dx$ be a contact form, in standard form, tangent to this chore. Let $\tilde{v}_0$, $\tilde{v}_1$ be two non singular vector fields in its kernel which both have the property that $\{0\} \times S^1$ is an attractive periodic orbit for them. We have proved in [5] that we could then find a smooth family of embedded tori $(S^1 \times S^1)_t$, which are boundaries of solid tori $(D^2 \times S^1)_t$ around $\{0\} \times S^1$, such that:

i) $ker\alpha$ is transverse to $(S^1 \times S^1)_t$

ii) a transverse direction of $\ker\alpha$ along $(S^1 \times S^1)_t$ can be extended into a smooth family v_t of vector fields in $ker\alpha$, with $v_0 = \tilde{v}_0$, $v_1 = \tilde{v}_1$, having all $\{0\} \times S^1$ as an attractive periodic orbit.

Let us choose $v_1 = \tilde{v}_1$ to be a vector field in $ker\alpha$ that has $\{0\} \times S^1$ as an attractive periodic orbit, in fact has $\{0\} \times S^1$ as ω-limit set in the torus $D^2 \times S^1$ and which, in addition, "turns well" near the chore. $\tilde{v}_1$ can be found explicitly.

Starting with a vector field v_0 that has $\{0\} \times S^1$ as attractive periodic orbit, but along which $ker\alpha$ might "not turn well", we want to replace progressively near the chore this vector field by another vector field $\tilde{v}$ that does not have new periodic orbits or a different ω-limit set, but such that $ker\alpha$ now turns well along its orbits near this chore.

The natural candidate for this is v_1, but we need to glue up v_0 and v_1. An appropriate modification of α has to be completed along this gluing-up procedure.

For this, we complete a diffeomorphism of $(S^1 \times S^1)_t$, with $t \in [0,1]$, with $\rho(t)S^1 \times S^1$, $\rho(t) = 1 - \epsilon t$, ϵ small. The trace of $ker\alpha$ on $(S^1 \times S^1)_t$ becomes then a non singular vector field $\tilde{w}_t$. We then modify the pull back of $ker\alpha$ into $ker\theta$ as follows: $ker\theta$ will be spanned by $\frac{\partial}{\partial\rho}$ and by $\tilde{w}_t$, where $\tilde{w}_t$ is derived from $w_t$after a forward rotation in a $\frac{\partial}{\partial\rho}$-transported frame by an angle $\mu(\rho)_t$, so that $ker\theta$ is again a contact structure (due to this forward rotation in this $\frac{\partial}{\partial\rho}$-transported frame). In this way, $ker\theta$ is defined outside a solid torus $(1-\epsilon)D^2 \times S^1$. v_1 points inwards this solid torus and $ker\alpha$ turns well along the positive orbits of v_1.

Starting from $(1-\epsilon)S^1 \times S^1$ with $\tilde{w}_1$, we know that the original trace w_1 of $ker\alpha$ on $(1-\epsilon)S^1 \times S^1$ is backwards with respect to $\tilde{w}_1$. Using the fact that $ker\alpha$, hence w_1, turns well, we can extend $\tilde{w}_1$ into a vector field that rotates forward very little along the positive orbits of v_1; it then follows on one hand that $ker\theta$ extends into a contact structure inside $(1-\epsilon)S^1 \times S^1$. Furthermore, w_1 eventually catches up with this $\tilde{w}_1$, thus allowing on a smaller solid torus to extend $ker\theta$ into $ker\alpha$.

We have now found a new vector field $\tilde{\tilde{v_1}}$ in the kernel of a new contact structure $\ker\theta$ such that the chore is the only ω-limit set in $D^2 \times S^1$ (it suffices to choose this $D^2 \times S^1$ so small as we start that the ω-limit set of v_0 in it is just the chore). Furthermore, this new contact structure turns well along this new vector field near this chore.

4. More hyperbolic orbits, J_∞ versus I_∞, V, "broken" very long/infinite $\pm v$-jumps.

1.More hyperbolic orbits.

In case there are more hyperbolic orbits, the argument generalizes. Indeed, the only novelty is that the stable manifold of one such hyperbolic orbit might intersect the unstable manifold of another one. However, this will happen along isolated flow-lines. Assuming that, along these flow-lines of intersection, the hyperbolic foliations of each periodic orbit are transversal to the hyperbolic foliations of the other periodic orbit, Proposition 1 extends and so do all Lemmas from 1 to 9.The compactness result follows.

2. J_∞ versus I_∞, V,.

Once the compactness result holds for I_∞, it also holds for J_∞ because Σ^- can be reduced to be, after modification of α into $\lambda\alpha$ and use of equation (1) and Lemma 1, a hyperbolic neighborhood of O_3 whose boundary is made of pieces of orbits of v and of pieces of orbits of ξ. Here, we are using the fact that $ker\alpha$, hence ξ -if the contact form is chosen appropriately in the contact structure- does not "turn well" along v near O_3, hence can be assumed to be basically v-transported (less than $\frac{\pi}{2}$ rotation in a v-transported frame); to the least, ξ can be assumed to be transverse to the stable and unstable manifolds of O_3 near O_3. It thus can be used to build entrance and exit sections to v near O_3. The hyperbolic neighborhood V of O_3 (equal to Σ^-) is then made of v-periodic orbits running from the entrance section to the exit section.

Trajectories of ξ spend very little time in V and thus, critical points of J_∞ are very close to critical points of I_∞; the compactness arguments thereby extend.

This shift from J_∞ to I_∞ should work under the more general circumstances in which ξ is transverse to the set S where v is "Anosov" and $ker\alpha$ is basically a foliation in the v-transport.

3."Broken" very long/infinite $\pm v$-jumps.

"Broken" infinite $\pm v$-jumps that are made of several pieces are limits of finite very long $\pm v$-jumps separated by tiny ξ-pieces. These tiny ξ-pieces can be considered as "free", non-degenerate. Therefore, we can enforce the (nearly) criticality conditions on them of the type $\alpha_{x_s}(d\gamma_s(\xi)) = 1 + o(1)$. If these are violated, we find decreasing directions which might decrease the already tiny ξ-piece rapidly or change the size of a very long $\pm v$-jump rapidly. However, the number of $\pm v$-jumps and ξ-pieces of the curve remains bounded, as it can only increase if we find decreasing normals from Γ_{2k} to Γ_{2k+2} as in the proof of Lemma 7. But then the related decrease is large and the process is not hindered by the size of a disappearing ξ-piece or a $\pm v$-jump becoming infinite.

Therefore, the compactness arguments, "localization " etc apply, until such a tiny ξ-piece disappears or the functional decreases substantially. We thus may assume that our very long/infinite $\pm v$-jumps are not "broken".

5. The use of (1).

(1) is a second order differential equation which can be used in order to rescale, adjust the rotation of ξ along the flow-lines of v. Given two points x_0, x_1 on the same v-orbit (running from x_0 to x_1), such that ξ at x_1 is derived from ξ at x_0 after a positive rotation (a rotation in the same direction than $ker\alpha$), (1) allows to redefine ξ and $ker\alpha$ between x_0 and x_1 so that ξ rotates monotonically from x_0 to x_1. This is completed with given boundary conditions at x_0 and x_1 for ξ and $ker\alpha$, see [9], Lemma 1 and Corollary 1, see also [9] for a specific example where the contact structure is adjusted between two tori. The alteration is a continuous function of x_0, x_1, ξ at x_0, x_1 and $ker\alpha$ at these points.

We thus need to define sections to v where the x_0s' and x_1s' will be taken and check that ξ at x_1 is derived from ξ at x_0 after a positive rotation. The discussion about Assumption (A) in section 3 above shows that we can assume, after slightly modifying v if needed near the attractive and repulsive orbits($ker\alpha$ and ξ turn less than $\frac{\pi}{2}$ near O_3), that this positive rotation either holds or can be introduced between corresponding points, x_0s' on the boundary of the solid torus near the repulsive orbit or on the exit set of v near O_3 on one hand and x_1s' on the entrance set of v near O_3 or on the boundary of a solid torus near the attractive orbit on the other hand. On the subset of ∂V made of v-portions of orbits, we then have two couples (x_0, x_1) and we need to be able to define ξ between the first x_1 and the second x_0 (going in the v-direction along the orbit) so that ξ is v-transported (up to some v-component) between these points. This again is a matter of reordering the rotation compatibly on both portions of the v-orbit.

Our claim about the use of (1) follows and it relies only on the fact that the ξ-rotation in a v-transported frame from the repulsive orbit to the attractive orbit is **globally**, not locally positive. This assumption is much weaker than Assumption (A).

6. A Conjecture.

In [4], we have related the homology which we have defined in [3] to the pull-back of the S^1- characteristic classes from PC^∞ to the rational homology of the space C_β (with some subset removed) under the classifying map for this S^1-action on C_β. This classifying map has been identified using the v-component b of the tangent vector to a curve x of C_β (which reads $\dot{x} = a\xi + bv$).

We then went in [4] one step further and related this homology to the S^1-equivariant homology of the loop space $\Lambda(M/v)$ of the space of v-orbits.

This conjectural relation has been "established" in the last part of [4] (section 3); it is heavily dependent on the behavior of the space of orbits M/v.

Our present results indicate strongly that, if v is a Morse-Smale vector field or a vector-field having a region where it is "Anosov" and a complement region where it behaves as a Morse-Smale vector field, then the spaces M/v, $\Lambda(M/v)$ can be defined appropriately and the correspondence between the homology for contact forms defined in [3] and the S^1-invariant rational homology of $\Lambda(M/v)$ can be established. The study of this conjecture under these more general assumptions is left to the future.

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Rutgers, The State University of New Jersey Department of Mathematics 110 Frelinghuysen Road Piscataway, NJ 08854-8019