RUTGERS UNIVERSITY

GRADUATE PROGRAM IN MATHEMATICS

Written Qualifying Examination

August 30, 2011, Day 1

This examination will be given in two three-hour sessions, one today and one tomorrow. At each session the examination will have two parts. Answer all three of the questions in Part I (numbered 1–3) and three of the six questions in Part II (numbered 4–9). If you work on more than three questions in Part II, indicate **clearly** which three should be graded. No additional credit will be given for more than three partial solutions. If no three questions are indicated, then the first three questions attempted in the order in which they appear in the examination book(s), and only those, will be the ones graded.

First Day—Part I: Answer each of the following three questions

1. First state both the monotone and dominated convergence theorems, and then prove that $f(x) = 1/\cosh(x)$ is Lebesgue integrable over $\mathbb{R}$.

2. Evaluate the integral

$$\int_C \frac{3z + e^{z^2}}{z^3} dz\,,$$

where z is the unit circle $\{z : |z| = 1\}$, oriented counterclockwise.

3. Let A and B be 7×7 matrices over the real numbers. Suppose

$$(A^2 + I)^2(A - 3I)^3 = 0 = (B^2 + I)^2(B - 3I)^3.$$

Suppose in addition that there exist nonzero elements $v, w \in \mathbb{R}^7$ such that if V is any proper subspace containing v, then $AV + V \neq V$ and, similarly, if W is any proper subspace containing w then $BW + W \neq W$. Prove that A is similar to B.

The exam continues on next page

First Day—Part II: Answer three of the following questions. If you work on more than three questions, indicate clearly which three should be graded.

4. Let μ be the Lebesgue measure on $\mathbb{R}^n$. For any $f \in L^1(\mu)$, prove that

$$\lim_{a \to 1} \int_{\mathbb{R}^n} |f(x) - a^n f(ax)| \mathrm{d}\mu = 0 \ .$$

5. Let $\mathbb{F}_2$ denote the field of 2 elements, and $\mathbb{F}_2[[t]]$ the power series ring in t. Determine how many nonisomorphic $\mathbb{F}_2[[t]]$-modules there are with 8 elements. Justify your answer.

6. Prove that the series $\Sigma_{n=-\infty}^{\infty} \frac{1}{(z-n)^2}$ converges to a meromorphic function on $\mathbb{C}$. Then prove that $F(z) = \frac{\pi^2}{\sin^2(\pi z)} - \Sigma_{n=-\infty}^{\infty} \frac{1}{(z-n)^2}$ is an entire function.

7. Suppose that $\{U_\alpha\}$ is a family of open subsets of $\mathbb{C}^2 \setminus 0$ whose union is all of $\mathbb{C}^2 \setminus 0$. Show that there is a finite number of indices $\alpha_1, ..., \alpha_n$ such that for every point v in $\mathbb{C}^2 \setminus 0$ there is a complex number z for which zv is in one of $U_{\alpha_1}, ..., U_{\alpha_n}$.

8. Let G be a group and H_1, H_2 be normal subgroups of G.
Prove that $G/(H_1 \cap H_2)$ is abelian if and only if G/H_1 and G/H_2 are abelian. Then give an example in which G/H_1 and G/H_2 are abelian, $H_1 \cap H_2$ is in the center of G, and G is not abelian.

9. Let $\Lambda(\mathbb{R}^3)$ be the Lebesgue σ-algebra of $\mathbb{R}^3$, and let $\mathcal{P}(\mathbb{R}^3)$ denote the power set of $\mathbb{R}^3$. Of course, $\Lambda(\mathbb{R}^3) \subset \mathcal{P}(\mathbb{R}^3)$. Show that λ, the 3-dimensional Lebesgue measure on $\Lambda(\mathbb{R}^3)$, cannot be extended to $\mathcal{P}(\mathbb{R}^3)$ [so that it is finitely additive and invariant under Euclidean congruence][1].
Hint: Use the Banach-Tarski paradox, which states the following:

Theorem: If U and V are arbitrary bounded open sets in $\mathbb{R}^n$, with $n \geq 3$, then there exists a $k \in \mathbb{N}$ and subsets $E_1, ..., E_k, F_1, ..., F_k$ of $\mathbb{R}^n$ such that
(i) $E_j \cap E_l = \emptyset$ whenever $j \neq l$, and $\bigcup_j E_j = U$;
(ii) $F_j \cap F_l = \emptyset$ whenever $j \neq l$, and $\bigcup_j F_j = V$;
(iii) $E_j \sim F_j$ for all $j = 1, ..., k$.

[1]content in square brakets are added later on for clarification.

(Here $\sim$ means Euclidean congruence, i.e. $A \sim B$ if A can be mapped into B by a combination of a translation, a rotation, and a reflection.)

Day 1 Exam End

RUTGERS UNIVERSITY

GRADUATE PROGRAM IN MATHEMATICS

Written Qualifying Examination

August 31, 2011, Day 2

This examination will be given in two three-hour sessions, today's being the second part. At each session the examination will have two parts. Answer all three of the questions in Part I (numbered 1–3) and three of the six questions in Part II (numbered 4–9). If you work on more than three questions in Part II, indicate **clearly** which three should be graded. No additional credit will be given for more than three partial solutions. If no three questions are indicated, then the first three questions attempted in the order in which they appear in the examination book(s), and only those, will be the ones graded.

Second Day—Part I: Answer each of the following three questions

1. Let μ be the Lebesgue measure on $\mathbb{R}$. Let

$$A := \left\{ f \in L^1(\mu) \ : \ \int_{\mathbb{R}} |f(x)|^2 \mathrm{d}\mu \geq 1 \right\},$$

$$B := \left\{ f \in L^1(\mu) \ : \ \int_{\mathbb{R}} |f(x)|^2 \mathrm{d}\mu \leq 1 \right\}.$$

(i) Is A a closed subset of $L^1(\mu)$? Justify your answer.

(ii) Is B a closed subset of $L^1(\mu)$? Justify your answer.

2. Exhibit a biholomorphic mapping (i.e., a holomorphic mapping $S \to D$ with a holomorphic inverse) from the open strip $S = \{z \in \mathbb{C} : 0 < \operatorname{Re} z < 1\}$ onto the open unit disc $D = \{z : |z| < 1\}$.

3. Let S_n denote the symmetric group on n elements.
(a) How many subgroups of order 33 are there in S_{13}? Why?
(b) How many subgroups of order 5 are there in S_6? Why?

The exam continues on next page

Second Day—Part II: Answer three of the following questions. If you work on more than three questions, indicate clearly which three should be graded.

4. Let I denote the interval $[0,1]$ and let λ denote Lebesgue measure on the Lebesgue σ-algebra $\Lambda(I)$. Let λ^2 denote the product measure on the product σ-algebra $\Lambda \otimes \Lambda$ for $I \times I$. Finally, set

$$f(x,y) = \frac{\sin(10x + 17y)}{1 - xy}$$

Find out whether the following three integrals (J_1, J_2 and J_{12}) are finite, and if so, whether there holds equality between them.

$J_1(f) = \int_I \left(\int_I f(x,y)\ d\lambda(y) \right) d\lambda(x)$,

$J_2(f) = \int_I \left(\int_I f(x,y)\ d\lambda(x) \right) d\lambda(y)$, $\qquad J_{12}(f) = \int_{I\times I} f(x,y)\ d\lambda^2$.

5. Let f, g be entire functions such that $f^2 = g^3$ (i.e., $f(z)^2 = g(z)^3$ for all $z \in \mathbb{C}$). Prove that there is an entire function h such that $h^3 = f$ and $h^2 = g$.

6. Let A be a matrix over the complex numbers. Suppose that the rational canonical form of A is a block diagonal matrix with blocks C_1, C_2 and C_3 where: C_1 is the companion matrix of x^2+1, C_2 is the companion matrix of x^4-1 and C_3 is the companion matrix of $x^6+2x^5+x^4-x^2-2x-1$. Find the Jordan canonical form of A.

7. Suppose that U is an open subset of the complex plane $\mathbb{C}$ with the property that for every holomorphic function $f : U \mapsto \mathbb{C}$ there exists a holomorphic function $F : U \mapsto \mathbb{C}$ such that $F' = f$. Prove that for every harmonic function $u : U \mapsto \mathbb{R}$ there exists a function $v : U \mapsto \mathbb{R}$ such that $u + iv$ is holomorphic.

8. Let B be a nondegenerate symmetric bilinear form on $\mathbb{R}^6$ with signature 0. Show that for every $v \in \mathbb{R}^6$ there is a $w \in \mathbb{R}^6$ [with $w \neq 0$] such that

$$B(v,w) = B(w,w) = 0.$$

9. Let X be a compact metric space and let $\mathcal{F}$ be any nonempty set of real valued functions on X that is uniformly bounded and equicontinuous. Is the function $g(x) = \sup\{\ f(x)\ :\ f \in \mathcal{F}\ \}$ necessarily continuous? Justify your answer.

Exam Day 2 End