Problems coming from Integrals

Dedicated to my most famous nephew

Victor H. Moll
Tulane University

May 30, 2010

Doron number 2

Tewodros Amdeberhan
Christoph Koutschan
Luis Medina
Eric Rowland
Xinyu Sun

General problem

Given a function

$$f : [a, b] \to \mathbb{R}$$

say something interesting about

$$I(f; a, b) := \int_a^b f(x)\, dx$$

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Ideal solution

`Input:` The function f and the parameters $\{a, b\}$

`Output:` The value of $\int_a^b f(x)\,dx$

and

An *interesting* fact about the answer.

We are far away from this

Ideal solution

`Input:` The function f and the parameters $\{a, b\}$

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General point of view

If you have an integral

I am interested in it

Before *Table look-up* there were tables

A small selection of tables of integrals

- D. Bierens de Haan, 1862, 1867
- M. Abramowitz and I. Stegun: now beautifully redone DLMF
- The Bateman manuscript project
- A. Apelblat, 1983
- I. S. Gradshteyn and I. M. Ryzhik (seventh edition, 2007)
- A. P. Prudnikov, Yu. A. Brychkov, O. I. Marichev (five volumes)
- A. Devoto and D. Duke: *Integrals useful in Feynman diagrams*
- R. Mathar: *Yet another Table of Integrals, 2010*

And now we have

- Mathematica, Maple, MatLab, MuPad
- TILU
- Sage

So many integrals
so little time

Sixth edition

-15-

3.247

✓1. $$\int_0^1 \frac{x^{\alpha-1}(1-x)^{n-1}}{1-\xi x^b}\,dx = (n-1)!\sum_{k=0}^{\infty}\frac{\xi^k}{(\alpha+kb),(\alpha+kb+1)\ldots(\alpha+kb+k-1)}$$ 596

n, drop

$[b>0, \quad |\xi|<1]$ AD (6704)

✓2. $$\int_0^\infty \frac{(1-x^p)\,x^{\nu-1}}{1-x^{np}}\,dx = \frac{\pi}{np}\sin\left(\frac{\pi}{n}\right)\operatorname{cosec}\frac{(p+\nu)\pi}{np}\operatorname{cosec}\frac{\pi\nu}{np}$$ 514

$[0<\operatorname{Re}\nu<(n-1)p]$ ET I 311(33)

3.248

✓1. [5] $$\int_0^\infty \frac{x^{\mu-1}\,dx}{\sqrt{1+x^\nu}} = \frac{1}{\nu}\mathrm{B}\left(\frac{\mu}{\nu},\frac{1}{2}-\frac{\mu}{\nu}\right)$$ (1) $[\operatorname{Re}\nu>\operatorname{Re}2\mu>0]$ BI (21)(9)

✓2. [5] $$\int_0^1 \frac{x^{2n+1}\,dx}{\sqrt{1-x^2}} = \frac{(2n)!!}{(2n+1)!!}$$ (2) BI (8)(14)

✓3. [5] $$\int_0^1 \frac{x^{2n}\,dx}{\sqrt{1-x^2}} = \frac{(2n-1)!!}{(2n)!!}\frac{\pi}{2}$$ (3) BI (8)(13)

✓4.[3] [6] $$\int_{-\infty}^\infty \frac{dx}{(1+x^2)\sqrt{4+3x^2}} = \frac{\pi}{3}$$ (4)

5.[9] $$\int_0^\infty \frac{dx}{(1+x^2)^{3/2}\left[1+\frac{4x^2}{3(1+x^2)^2}+\left(1+\frac{4x^2}{3(1+x^2)^2}\right)^{1/2}\right]^{1/2}} = \frac{\pi}{2\sqrt{6}}$$

3.248.5

$$\int_0^\infty \frac{dx}{(1+x^2)^{3/2}\sqrt{\left[\varphi(x)+\sqrt{\varphi(x)}\right]}} = \frac{\pi}{2\sqrt{6}}$$

with

$$\varphi(x) = 1 + \frac{4x^2}{3(1+x^2)^2}$$

Beautiful but incorrect

Problem: evaluate it

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Seventh edition

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1. $\int_0^\infty \frac{x^{\mu-1}\,dx}{\sqrt{1+x^\nu}} = \frac{1}{\nu} \mathrm{B}\left(\frac{\mu}{\nu}, \frac{1}{2} - \frac{\mu}{\nu}\right)$ ⑥ $[\operatorname{Re}\nu > \operatorname{Re}2\mu > 0]$ BI (21)(9)

2. $\int_0^1 \frac{x^{2n+1}\,dx}{\sqrt{1-x^2}} = \frac{(2n)!!}{(2n+1)!!}$ ⑥ BI (8)(14)

3. $\int_0^1 \frac{x^{2n}\,dx}{\sqrt{1-x^2}} = \frac{(2n-1)!!}{(2n)!!}\frac{\pi}{2}$ ⑥ BI (8)(13)

4.[3] $\int_{-\infty}^\infty \frac{dx}{(1+x^2)\sqrt{4+3x^2}} = \frac{\pi}{3}$ ⑦

6.* $\int_{-\infty}^\infty \frac{dx}{(1+x^2)^2\sqrt{b+ax^2}} = \begin{cases} \frac{2}{\sqrt{b-a}} \arctan\left(\sqrt{\frac{b}{a}-1}\right) & \text{if } a < b \\ \frac{2}{\sqrt{a}} & \text{if } a = b \\ \frac{1}{\sqrt{a-b}} \ln\left(\frac{\sqrt{a}+\sqrt{a-b}}{\sqrt{a}-\sqrt{a-b}}\right) & \text{if } a > b \end{cases}$ ⑦

3.249

1.[0] $\int_0^\infty \frac{dx}{(x^2+a^2)^n} = \frac{(2n-3)!!}{2\cdot(2n-2)!!}\frac{\pi}{a^{2n-1}}$ ⑥ ⑦ FI II 743

2.[9] $\int_0^a (a^2-x^2)^{n-\frac{1}{2}}\,dx = a^{2n}\frac{(2n-1)!!}{2(2n)!!}\pi.$ ⑥ FI II 156

3. $\int^1 \frac{(1-x^2)^n\,dx}{(\ldots)^{n+1}} = 2^{n+1}Q_n(a)$ EH II 181(31)

Why did we pick this formula?

Expand $\sqrt{a+\sqrt{1+c}}$ as a Taylor series in c:

$$\sqrt{a+\sqrt{1+c}} = \sqrt{a+1}\left(1+\sum_{k=1}^{\infty}\frac{(-1)^{k-1}P_{k-1}(a)}{k2^{k+1}(a+1)^k}\,c^k\right)$$

$$\int_0^{\infty}\frac{dx}{(x^4+2ax^2+1)^{m+1}} = \frac{\pi}{2}\left[2(a+1)\right]^{-(m+1/2)}\,P_m(a)$$

$$P_m(a) := \sum_{j=0}^{m} d_{j,m}a^j$$

$$d_{j,m} = 2^{-2m}\sum_{k=j}^{m}2^k\binom{2m-2k}{m-k}\binom{m+k}{m}\binom{k}{j}$$

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An extension

Problem:
Find a proof of the double square root expansion that extends to

$$\sqrt{a+\sqrt{b+\sqrt{1+c}}}$$

The conjectured formula involves homogenizations of the P_m.

Perhaps some geometry should be involved.

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The original formula

$$d_{r,m} = \sum_{j=0}^{r} \sum_{s=0}^{m-r} \sum_{k=s+r}^{m} \frac{(-1)^{k-r-s}}{2^{3k}} \binom{2k}{k} \binom{2m+1}{2s+2j} \binom{m-s-j}{m-k} \binom{s+j}{j} \binom{k-s-j}{r-j}$$

Make a table of values to see that $d_{r,m} > 0$.

A pretty identity

Evaluate both expressions for $P_m(1)$ to get

$$\sum_{k=0}^{m} 2^{-2k} \binom{2k}{k} \binom{2m-k}{m} = \sum_{k=0}^{m} 2^{-2k} \binom{2k}{k} \binom{2m+1}{2k}$$

- Elementary proof
- Complex analytic proof
- Hypergeometric style proof
- Of course: WZ proof
- Combinatorial proof: NO.

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How can you not buy this?

A=B

Marko Petkovšek
University of Ljubljana
Ljubljana, Slovenia

Herbert S. Wilf
University of Pennsylvania
Philadelphia, PA, USA

Doron Zeilberger
Temple University
Philadelphia, PA, USA

My first Doron-contact

TEMPLE UNIVERSITY
A Commonwealth University

College of Arts and Sciences
Department of Mathematics

Computer Building 038-16
Philadelphia, Pennsylvania 19122

May 13, 1996

Dear Prof. Holl,

Thanks for your letter.
Indeed, the triple sum that you had does not seem to have closed form in both m and p.
For a fixed $m-p=L$, it does, but as L gets bigger, the "closed form"

The $d_{j,m}$ take you to many places

Theorem

The sequence $d_{j,m} : 0 \leq j \leq m$ is unimodal.
(G. Boros, V. M.)

This should be my Doron number 1 paper

Theorem

The sequence $d_{j,m} : 0 \leq j \leq m$ is logconcave.
(P. Paule-M. Kauers)

There is no (need according to Doron) purely human proof

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Problem: The sequence is ∞-logconcave.

The $d_{j,m}$ take you to many places. Continuation

Theorem

G. Boros - J. Shallit - V.M. (2000) The 2-adic valuation of $d_{1,m}$ is given by

$$\nu_2(d_{1,m}) = 1 - 2m + \nu_2\left(\binom{m+1}{2}\right) + s_2(m)$$

$s_2(m) =$ *number of* 1 *in the binary expansion of* m.

The function $s_2(m)$ is ubiquotous:

Theorem

T. Lengyel (1994), S. De Wannemacher (2005)

$$\nu_2(S(2^n, k)) = s_2(k) - 1$$

$S(n, k)$ is the Stirling number of second kind

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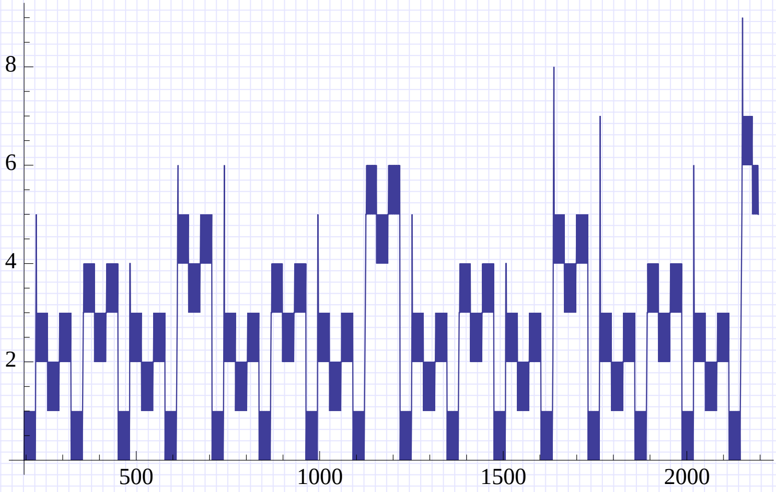
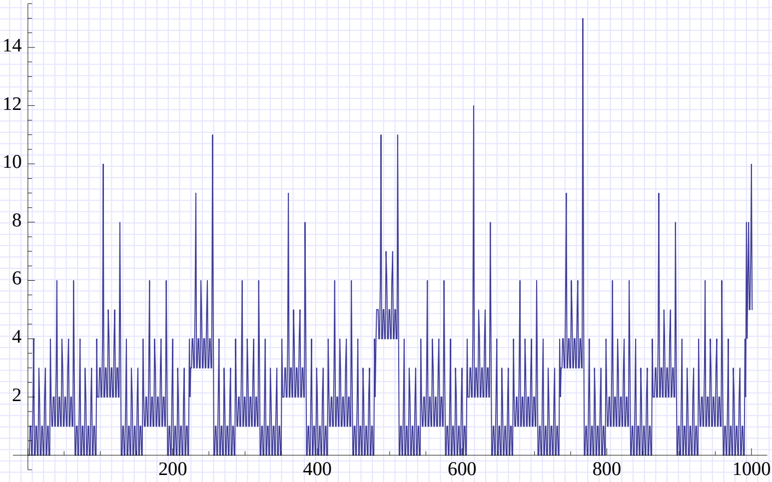
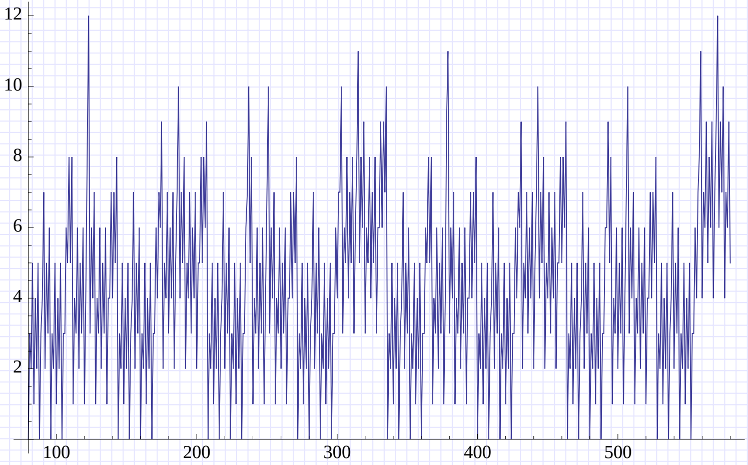
Nice Stirling numbers



Hidden structure Stirling numbers



Ugly Stirling numbers



Some loggamma integrals

$$L_1 = \int_0^1 \ln\Gamma(q)\,dq = \ln\sqrt{2\pi} \qquad \text{Euler}$$

Some loggamma integrals. Continuation

Lerch's formula

$$\frac{\partial}{\partial z}\zeta(z,q)\Big|_{z=0} = \ln\Gamma(q) - \ln\sqrt{2\pi}$$

combine with the Fourier expansion of Hurwitz zeta:

$$L_2 := \int_0^1 \ln^2\Gamma(q)\,dq$$
$$= \frac{\gamma^2}{12} + \frac{\pi^2}{48} + \frac{\gamma L_1}{3} + \frac{4}{3}\gamma L_1 - (\gamma + 2L_1)\frac{\gamma'(2)}{\pi^2} + \frac{\zeta''(2)}{2\pi^2}$$

O. Espinosa and V.M. (2002)

Some loggamma integrals. Continuation

$$L_3 := \int_0^1 \ln^3 \Gamma(q)\, dq$$

????????????????????

LLL for L_3

Evaluations with LLL:
given a set of (symbolic) real numbers and a decimal expansion ρ

it computes the best expression for ρ as a linear combination of the basis elements

Compute L_2 numerically to high precision

Take as basis all products pq

p polynomial in π, $\ln 2$, $\ln \pi$, γ deg $p \leq 2$

q is either 1, $\zeta'(2)$ or $\zeta''(2)$

30 elements

LLL gives the analytic expression for L_2

This did not work for L_3.

Basis includes $\zeta'''(2)$ and $\zeta(3)$.

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Some logsine integrals

$$\Gamma(x)\Gamma(1-x) = \frac{\pi}{\sin \pi x}$$

leads one to work on

$$S_n := (-1)^n \int_0^1 \ln^n(\sin \pi x)\, dx$$

$$S_2 = \frac{\pi^2}{12} + \ln^2 2$$

Theorem

S_n is a homogeneous polynomial in $z_0 := \ln 2$, $z_1 = \pi$ and $z_j = \zeta(j)^{1/j}$

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Logsine integrals: continuation

$$U_n := \sum_{k=0}^{n} \binom{n}{k} z_0^{n-k} S_k$$

Problem: U_n is obtained from S_n by replacing $z_0 = \ln 2$ by $\ln(2\pi)$

$$\begin{aligned} S_2 &= \frac{\pi^2}{12} + \ln^2 2 \\ &\mapsto \frac{\pi^2}{12} + (\ln 2 + \ln \pi)^2 \\ &= \frac{\pi^2}{12} + \ln^2 2 + 2 \ln 2 \ln \pi + \ln^2 \pi. \\ &= U_2 \end{aligned}$$

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$$\int_0^1 \ln^3 \Gamma(x)\,dx + 3\int_0^1 \ln^2 \Gamma(x)\,\ln\Gamma(x)\,dx = \frac{1}{8}\left(\pi^2 \ln(2\pi) + 4\ln^3(2\pi) + 6\zeta(3)\right)$$

Problem: How to separate these two integrals?

Logsine integrals: continuation

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$$S_n := (-1)^n \int_0^1 \ln^n(\sin \pi x)\, dx$$

$$S_n = \sum_{j=0}^{n} \alpha_{n,j} \ln^j 2$$

$$S_3 = \ln^3 2 + \frac{1}{4}\pi^2 \ln 2 + \frac{3}{2}\zeta(3)$$

$$K_n := \mathbb{Q}(\zeta(2), \zeta(3), \cdots, \zeta(n))$$

Problem:

$$\alpha_{n,j} \in K_n$$

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Logsine integrals: continuation

$M_d :=$ set of all monomials in $\zeta(j)$ of weight d

degree $\zeta(j) = j$

$$\alpha_{n,n-j} = (n-j+1)_j \sum_{m \in M_d} C_m m$$

for some $C_m \in \mathbb{Q}$ and π has even powers.

Problem: $C(m)$ is multiplicative:

$$C\left(\pi^{i_1}\,\zeta(3)^{i_2}\,\zeta(5)^{i_3}\cdots\right) = C\left(\pi^{i_1}\right) C\left(\zeta(3)^{i_2}\right) C\left(\zeta(5)^{i_3}\right)\cdots$$

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A birthday present

Stefan Boettner, Tulane thesis, April 2010
Automatic extensions of the Risch-Norman algorithm for integration
Based on the notion of differential field

$$\int \frac{E(x)\,dx}{x(1-x^2)\,K(k)} = \log x + \log K(x)$$

$$\int \left[2x^4 \mathrm{Ai}(x^2) + \mathrm{Ai}'(x^2)\right]\,dx = x\mathrm{Ai}'(x^2)$$

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Stefan Boettner, Tulane thesis, April 2010
Automatic extensions of the Risch-Norman algorithm for integration
Based on the notion of differential field

$$\int \frac{E(x)\,dx}{x(1-x^2)\,K(k)} = \log x + \log K(x)$$

$$\int \left[2x^4 \mathrm{Ai}(x^2) + \mathrm{Ai}'(x^2)\right]\, dx = x\mathrm{Ai}'(x^2)$$

The moral of Stefan's thesis

I still believe that humans are useful in Mathematics

but some of my children do not

That is progress

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Happy Birthday Doron