Descent sets of cyclic permutations

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The permutation 4217536 can be *realized* by the shift on 3 letters, because taking $w = 2102212210\ldots \in \mathcal{W}_3$,

$$\left.\begin{array}{rlc} w &= 2102212210\ldots & 4 \\ \Sigma(w) &= 102212210\ldots & 2 \\ \Sigma^2(w) &= 02212210\ldots & 1 \\ \Sigma^3(w) &= 2212210\ldots & 7 \\ \Sigma^4(w) &= 212210\ldots & 5 \\ \Sigma^5(w) &= 12210\ldots & 3 \\ \Sigma^6(w) &= 2210\ldots & 6 \end{array}\right\} \text{lexicographic order of the shifted words}$$

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- $N(\pi)$ is related to the number of descents of the cycle $(\pi_1, \pi_2, \ldots, \pi_n)$.

Computation of $N(\pi)$

Let $\hat{\pi}$ be the cycle $(\pi_1, \pi_2, \ldots, \pi_n)$ with the entry π_1 replaced with $\star$.

Example

$$\pi = 892364157 \rightsquigarrow (8,9,2,3,6,4,1,5,7) \rightsquigarrow 536174892 \rightsquigarrow 536174{\star}92 = \hat{\pi}$$

$$\mathrm{des}(536174{\star}92) = 4$$

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Theorem

$$N(\pi) = 1 + \mathrm{des}(\hat{\pi})$$

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So I became interested in the distribution of descent sets in cyclic permutations.

Descent sets of 5-cycles

$\mathcal{C}_5$	
$(1,2,3,4,5) = 2345{\cdot}1$	
$(2,1,3,4,5) = 3{\cdot}145{\cdot}2$	
$(3,2,1,4,5) = 4{\cdot}125{\cdot}3$	
$(4,3,2,1,5) = 5{\cdot}1234$	
$(1,3,2,4,5) = 34{\cdot}25{\cdot}1$	
$(1,4,3,2,5) = 45{\cdot}23{\cdot}1$	
$(3,1,2,4,5) = 24{\cdot}15{\cdot}3$	
$(3,1,4,2,5) = 45{\cdot}123$	
$(4,3,1,2,5) = 25{\cdot}134$	
$(1,2,4,3,5) = 245{\cdot}3{\cdot}1$	
$(2,4,1,3,5) = 345{\cdot}12$	
$(4,1,2,3,5) = 235{\cdot}14$	

$\mathcal{C}_5$	
$(2,3,1,4,5) = 4{\cdot}3{\cdot}15{\cdot}2$	
$(2,4,3,1,5) = 5{\cdot}4{\cdot}13{\cdot}2$	
$(4,2,3,1,5) = 5{\cdot}3{\cdot}124$	
$(1,4,2,3,5) = 4{\cdot}35{\cdot}2{\cdot}1$	
$(2,1,4,3,5) = 4{\cdot}15{\cdot}3{\cdot}2$	
$(2,3,4,1,5) = 5{\cdot}34{\cdot}12$	
$(3,4,2,1,5) = 5{\cdot}14{\cdot}23$	
$(4,2,1,3,5) = 3{\cdot}15{\cdot}24$	
$(1,3,4,2,5) = 35{\cdot}4{\cdot}2{\cdot}1$	
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$(3,2,1,4,5) = 4\cdot 125\cdot 3$	$3\cdot 124$
$(4,3,2,1,5) = 5\cdot 1234$	$4\cdot 123$
$(1,3,2,4,5) = 34\cdot 25\cdot 1$	$13\cdot 24$
$(1,4,3,2,5) = 45\cdot 23\cdot 1$	$14\cdot 23$
$(3,1,2,4,5) = 24\cdot 15\cdot 3$	$23\cdot 14$
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$(2,4,3,1,5) = 5\cdot 4\cdot 13\cdot 2$	$4\cdot 2\cdot 13$
$(4,2,3,1,5) = 5\cdot 3\cdot 124$	$4\cdot 3\cdot 12$
$(1,4,2,3,5) = 4\cdot 35\cdot 2\cdot 1$	$3\cdot 24\cdot 1$
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Main theorem

Let $D(\pi) = \{i \,:\, 1 \leq i \leq n-1,\ \pi(i) > \pi(i+1)\}$.

Theorem

For every n there is a bijection $\varphi : \mathcal{C}_{n+1} \to \mathcal{S}_n$ *such that if* $\pi \in \mathcal{C}_{n+1}$ *and* $\sigma = \varphi(\pi)$*, then*

$$D(\pi) \cap [n-1] = D(\sigma).$$

The bijection $\varphi : \mathcal{C}_{n+1} \to \mathcal{S}_n$. First step

Given $\pi \in \mathcal{C}_{n+1}$, write it in cycle form with $n+1$ at the end:

$$\pi = (11, 4, 10, 1, 7, 16, 9, 3, 5, 12, 20, 2, 6, 14, 18, 8, 13, 19, 15, 17, 21) \in \mathcal{C}_{21}$$

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Delete $n+1$ and split at the "left-to-right maxima":

$$\sigma = (\underline{11}, 4, 10, 1, 7)(\underline{16}, 9, 3, 5, 12)(\underline{20}, 2, 6, 14, 18, 8, 13, 19, 15, 17) \in \mathcal{S}_{20}.$$

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This map $\pi \mapsto \sigma$ is a bijection, but unfortunately it does not always preserve the descent set:

$$\pi(7) = 16 > \pi(8) = 13 \quad \text{but} \quad \sigma(7) = 11 < \sigma(8) = 13.$$

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We say that the pair $\{7, 8\}$ is *bad*. We will fix the bad pairs.

The bijection $\varphi : \mathcal{C}_{n+1} \longrightarrow \mathcal{S}_n$. Fixing bad pairs

$$\begin{aligned} \pi &= (11,4,10,1,7,16,9,3,5,12,20,2,6,14,18,8,13,19,15,17,21) \\ \sigma &= (11,4,10,1,7)(16,9,3,5,12)(20,2,6,14,18,8,13,19,15,17) \end{aligned}$$

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For each but the last cycle of σ, from left to right:

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$\{7, 6\}$ and $\{7, 8\}$ are bad; and $\sigma(6) = 14 > 13 = \sigma(8) \Rightarrow \ \varepsilon := -1$.

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  3. $z :=$ new rightmost entry of the cycle.

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$z := 6$. $\{6, 5\}$ is bad. $\varepsilon := -1$.

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  2. If the elements preceding the last switched entries have consecutive values, switch them. Repeat 2.
  3. $z :=$ new rightmost entry of the cycle.

$$\pi = (11, 4, 10, 1, 7, 16, 9, 3, 5, 12, 20, 2, 6, 14, 18, 8, 13, 19, 15, 17, 21)$$
$$\sigma = (11, 4, 10, 3, 5)(16, 9, 2, 6, 12)(20, 1, 7, 14, 18, 8, 13, 19, 15, 17)$$

$z := 6$. $\varepsilon := -1$.

Switch 6 and 5. Switch 2 and 3.

The bijection $\varphi : \mathcal{C}_{n+1} \to \mathcal{S}_n$. Fixing bad pairs

For each but the last cycle of σ, from left to right:

- $z :=$ rightmost entry of the cycle.
  If $\{z, z-1\}$ or $\{z, z+1\}$ are bad, let $\varepsilon = \pm 1$ be such that $\{z, z+\varepsilon\}$ is bad and $\sigma(z+\varepsilon)$ is largest.
- Repeat for as long as $\{z, z+\varepsilon\}$ is bad:
  1. Switch z and $z+\varepsilon$ (in the cycle form of σ).
  2. If the elements preceding the last switched entries have consecutive values, switch them. Repeat 2.
  3. $z :=$ new rightmost entry of the cycle.

$$\pi = (11, 4, 10, 1, 7, 16, 9, 3, 5, 12, 20, 2, 6, 14, 18, 8, 13, 19, 15, 17, 21)$$
$$\sigma = (11, 4, 10, 3, 5)(16, 9, 2, 6, 12)(20, 1, 7, 14, 18, 8, 13, 19, 15, 17)$$

$z := 6$. $\varepsilon := -1$.

Switch 6 and 5. Switch 2 and 3. Switch 10 and 9.

The bijection $\varphi : \mathcal{C}_{n+1} \to \mathcal{S}_n$. Fixing bad pairs

For each but the last cycle of σ, from left to right:

- $z :=$ rightmost entry of the cycle.
  If $\{z, z-1\}$ or $\{z, z+1\}$ are bad, let $\varepsilon = \pm 1$ be such that $\{z, z+\varepsilon\}$ is bad and $\sigma(z+\varepsilon)$ is largest.
- Repeat for as long as $\{z, z+\varepsilon\}$ is bad:
  1. Switch z and $z+\varepsilon$ (in the cycle form of σ).
  2. If the elements preceding the last switched entries have consecutive values, switch them. Repeat 2.
  3. $z :=$ new rightmost entry of the cycle.

$$\pi = (11, 4, 10, 1, 7, 16, 9, 3, 5, 12, 20, 2, 6, 14, 18, 8, 13, 19, 15, 17, 21)$$
$$\sigma = (11, \underline{4}, 9, 3, 5)(\underline{16}, 10, 2, 6, 12)(20, 1, 7, 14, 18, 8, 13, 19, 15, 17)$$

$z := 6$. $\varepsilon := -1$.

Switch 6 and 5. Switch 2 and 3. Switch 10 and 9.

The bijection $\varphi : \mathcal{C}_{n+1} \to \mathcal{S}_n$. Fixing bad pairs

For each but the last cycle of σ, from left to right:

- $z :=$ rightmost entry of the cycle.
  If $\{z, z-1\}$ or $\{z, z+1\}$ are bad, let $\varepsilon = \pm 1$ be such that $\{z, z+\varepsilon\}$ is bad and $\sigma(z+\varepsilon)$ is largest.
- Repeat for as long as $\{z, z+\varepsilon\}$ is bad:
  1. Switch z and $z+\varepsilon$ (in the cycle form of σ).
  2. If the elements preceding the last switched entries have consecutive values, switch them. Repeat 2.
  3. $z :=$ new rightmost entry of the cycle.

$$\pi = (11, 4, 10, 1, 7, 16, 9, 3, 5, 12, 20, 2, 6, 14, 18, 8, 13, 19, 15, 17, 21)$$
$$\sigma = (11, 4, 9, 3, 5)(16, 10, 2, 6, 12)(20, 1, 7, 14, 18, 8, 13, 19, 15, 17)$$

$z := 5.$ $\varepsilon := -1.$

The bijection $\varphi : \mathcal{C}_{n+1} \to \mathcal{S}_n$. Fixing bad pairs

For each but the last cycle of σ, from left to right:

- $z :=$ rightmost entry of the cycle.
  If $\{z, z-1\}$ or $\{z, z+1\}$ are bad, let $\varepsilon = \pm 1$ be such that $\{z, z+\varepsilon\}$ is bad and $\sigma(z+\varepsilon)$ is largest.
- Repeat for as long as $\{z, z+\varepsilon\}$ is bad:
  1. Switch z and $z+\varepsilon$ (in the cycle form of σ).
  2. If the elements preceding the last switched entries have consecutive values, switch them. Repeat 2.
  3. $z :=$ new rightmost entry of the cycle.

$$\pi = (11, 4, 10, 1, 7, 16, 9, 3, 5, 12, 20, 2, 6, 14, 18, 8, 13, 19, 15, 17, 21)$$
$$\sigma = (11, 4, 9, 3, 5)(16, 10, 2, 6, 12)(20, 1, 7, 14, 18, 8, 13, 19, 15, 17)$$

$z := 5$. $\{5, 4\}$ is OK, so we move on to the second cycle.

The bijection $\varphi : \mathcal{C}_{n+1} \to \mathcal{S}_n$. Fixing bad pairs

For each but the last cycle of σ, from left to right:

- $z :=$ rightmost entry of the cycle.
  If $\{z, z-1\}$ or $\{z, z+1\}$ are bad, let $\varepsilon = \pm 1$ be such that $\{z, z+\varepsilon\}$ is bad and $\sigma(z+\varepsilon)$ is largest.
- Repeat for as long as $\{z, z+\varepsilon\}$ is bad:
  1. Switch z and $z+\varepsilon$ (in the cycle form of σ).
  2. If the elements preceding the last switched entries have consecutive values, switch them. Repeat 2.
  3. $z :=$ new rightmost entry of the cycle.

$$\pi = (11, 4, 10, 1, 7, 16, 9, 3, 5, 12, 20, 2, 6, 14, 18, 8, 13, 19, 15, 17, 21)$$
$$\sigma = (11, 4, 9, 3, 5)(16, 10, 2, 6, 12)(20, 1, 7, 14, 18, 8, 13, 19, 15, 17)$$

$z := 12$.

The bijection $\varphi : \mathcal{C}_{n+1} \to \mathcal{S}_n$. Fixing bad pairs

For each but the last cycle of σ, from left to right:

- $z :=$ rightmost entry of the cycle.
  If $\{z, z-1\}$ or $\{z, z+1\}$ are bad, let $\varepsilon = \pm 1$ be such that $\{z, z+\varepsilon\}$ is bad and $\sigma(z+\varepsilon)$ is largest.
- Repeat for as long as $\{z, z+\varepsilon\}$ is bad:
  1. Switch z and $z+\varepsilon$ (in the cycle form of σ).
  2. If the elements preceding the last switched entries have consecutive values, switch them. Repeat 2.
  3. $z :=$ new rightmost entry of the cycle.

$$\pi = (11, 4, 10, 1, 7, 16, 9, 3, 5, 12, 20, 2, 6, 14, 18, 8, 13, 19, 15, 17, 21)$$
$$\sigma = (11, 4, 9, 3, 5)(16, 10, 2, 6, 12)(20, 1, 7, 14, 18, 8, 13, 19, 15, 17)$$

$z := 12$. $\{12, 11\}$ is OK but $\{12, 13\}$ is bad $\Rightarrow \varepsilon := 1$.

The bijection $\varphi : \mathcal{C}_{n+1} \to \mathcal{S}_n$. Fixing bad pairs

For each but the last cycle of σ, from left to right:

- $z :=$ rightmost entry of the cycle.
  If $\{z, z-1\}$ or $\{z, z+1\}$ are bad, let $\varepsilon = \pm 1$ be such that $\{z, z+\varepsilon\}$ is bad and $\sigma(z+\varepsilon)$ is largest.
- Repeat for as long as $\{z, z+\varepsilon\}$ is bad:
  1. Switch z and $z+\varepsilon$ (in the cycle form of σ).
  2. If the elements preceding the last switched entries have consecutive values, switch them. Repeat 2.
  3. $z :=$ new rightmost entry of the cycle.

$$\pi = (11, 4, 10, 1, 7, 16, 9, 3, 5, 12, 20, 2, 6, 14, 18, 8, 13, 19, 15, 17, 21)$$
$$\sigma = (11, 4, 9, 3, 5)(16, 10, 2, 6, 12)(20, 1, 7, 14, 18, 8, 13, 19, 15, 17)$$

$z := 12$. $\varepsilon := 1$.

Switch 12 and 13.

The bijection $\varphi : \mathcal{C}_{n+1} \longrightarrow \mathcal{S}_n$. Fixing bad pairs

For each but the last cycle of σ, from left to right:

- $z :=$ rightmost entry of the cycle.
  If $\{z, z-1\}$ or $\{z, z+1\}$ are bad, let $\varepsilon = \pm 1$ be such that $\{z, z+\varepsilon\}$ is bad and $\sigma(z+\varepsilon)$ is largest.
- Repeat for as long as $\{z, z+\varepsilon\}$ is bad:
  1. Switch z and $z+\varepsilon$ (in the cycle form of σ).
  2. If the elements preceding the last switched entries have consecutive values, switch them. Repeat 2.
  3. $z :=$ new rightmost entry of the cycle.

$$\pi = (11, 4, 10, 1, 7, 16, 9, 3, 5, 12, 20, 2, 6, 14, 18, 8, 13, 19, 15, 17, 21)$$
$$\sigma = (11, 4, 9, 3, 5)(16, 10, 2, 6, 13)(20, 1, 7, 14, 18, 8, 12, 19, 15, 17)$$

$z := 12$. $\varepsilon := 1$.
Switch 12 and 13.

The bijection $\varphi : \mathcal{C}_{n+1} \to \mathcal{S}_n$. Fixing bad pairs

For each but the last cycle of σ, from left to right:

- $z :=$ rightmost entry of the cycle.
  If $\{z, z-1\}$ or $\{z, z+1\}$ are bad, let $\varepsilon = \pm 1$ be such that $\{z, z+\varepsilon\}$ is bad and $\sigma(z+\varepsilon)$ is largest.
- Repeat for as long as $\{z, z+\varepsilon\}$ is bad:
  1. Switch z and $z+\varepsilon$ (in the cycle form of σ).
  2. If the elements preceding the last switched entries have consecutive values, switch them. Repeat 2.
  3. $z :=$ new rightmost entry of the cycle.

$$\pi = (11, 4, 10, 1, 7, 16, 9, 3, 5, 12, 20, 2, 6, 14, 18, 8, 13, 19, 15, 17, 21)$$
$$\sigma = (11, 4, 9, 3, 5)(16, 10, 2, \underline{6}, 13)(20, 1, 7, 14, 18, \underline{8}, 12, 19, 15, 17)$$

$z := 12$. $\varepsilon := 1$.

Switch 12 and 13.

The bijection $\varphi : \mathcal{C}_{n+1} \to \mathcal{S}_n$. Fixing bad pairs

For each but the last cycle of σ, from left to right:

- $z :=$ rightmost entry of the cycle.
  If $\{z, z-1\}$ or $\{z, z+1\}$ are bad, let $\varepsilon = \pm 1$ be such that $\{z, z+\varepsilon\}$ is bad and $\sigma(z+\varepsilon)$ is largest.
- Repeat for as long as $\{z, z+\varepsilon\}$ is bad:
  1. Switch z and $z+\varepsilon$ (in the cycle form of σ).
  2. If the elements preceding the last switched entries have consecutive values, switch them. Repeat 2.
  3. $z :=$ new rightmost entry of the cycle.

$$\pi = (11, 4, 10, 1, 7, 16, 9, 3, 5, 12, 20, 2, 6, 14, 18, 8, 13, 19, 15, 17, 21)$$
$$\sigma = (11, 4, 9, 3, 5)(16, 10, 2, 6, 13)(20, 1, 7, 14, 18, 8, 12, 19, 15, 17)$$

$z := 13$. $\varepsilon := 1$.

The bijection $\varphi : \mathcal{C}_{n+1} \to \mathcal{S}_n$. Fixing bad pairs

For each but the last cycle of σ, from left to right:

- $z :=$ rightmost entry of the cycle.
  If $\{z, z-1\}$ or $\{z, z+1\}$ are bad, let $\varepsilon = \pm 1$ be such that $\{z, z+\varepsilon\}$ is bad and $\sigma(z+\varepsilon)$ is largest.
- Repeat for as long as $\{z, z+\varepsilon\}$ is bad:
  1. Switch z and $z+\varepsilon$ (in the cycle form of σ).
  2. If the elements preceding the last switched entries have consecutive values, switch them. Repeat 2.
  3. $z :=$ new rightmost entry of the cycle.

$$\pi = (11, 4, 10, 1, 7, 16, 9, 3, 5, 12, 20, 2, 6, 14, 18, 8, 13, 19, 15, 17, 21)$$
$$\sigma = (11, 4, 9, 3, 5)(16, 10, 2, 6, 13)(20, 1, 7, 14, 18, 8, 12, 19, 15, 17)$$

$z := 13$. $\{13, 14\}$ is bad. $\varepsilon := 1$.

The bijection $\varphi : \mathcal{C}_{n+1} \to \mathcal{S}_n$. Fixing bad pairs

For each but the last cycle of σ, from left to right:

- $z :=$ rightmost entry of the cycle.
  If $\{z, z-1\}$ or $\{z, z+1\}$ are bad, let $\varepsilon = \pm 1$ be such that $\{z, z+\varepsilon\}$ is bad and $\sigma(z+\varepsilon)$ is largest.
- Repeat for as long as $\{z, z+\varepsilon\}$ is bad:
  1. Switch z and $z+\varepsilon$ (in the cycle form of σ).
  2. If the elements preceding the last switched entries have consecutive values, switch them. Repeat 2.
  3. $z :=$ new rightmost entry of the cycle.

$$\pi = (11, 4, 10, 1, 7, 16, 9, 3, 5, 12, 20, 2, 6, 14, 18, 8, 13, 19, 15, 17, 21)$$
$$\sigma = (11, 4, 9, 3, 5)(16, 10, 2, 6, 13)(20, 1, 7, 14, 18, 8, 12, 19, 15, 17)$$

$z := 13$. $\varepsilon := 1$.

Switch 13 and 14.

The bijection $\varphi : \mathcal{C}_{n+1} \to \mathcal{S}_n$. Fixing bad pairs

For each but the last cycle of σ, from left to right:

- $z :=$ rightmost entry of the cycle.
  If $\{z, z-1\}$ or $\{z, z+1\}$ are bad, let $\varepsilon = \pm 1$ be such that $\{z, z+\varepsilon\}$ is bad and $\sigma(z+\varepsilon)$ is largest.
- Repeat for as long as $\{z, z+\varepsilon\}$ is bad:
  1. Switch z and $z+\varepsilon$ (in the cycle form of σ).
  2. If the elements preceding the last switched entries have consecutive values, switch them. Repeat 2.
  3. $z :=$ new rightmost entry of the cycle.

$$\pi = (11, 4, 10, 1, 7, 16, 9, 3, 5, 12, 20, 2, 6, 14, 18, 8, 13, 19, 15, 17, 21)$$
$$\sigma = (11, 4, 9, 3, 5)(16, 10, 2, 6, 14)(20, 1, 7, 13, 18, 8, 12, 19, 15, 17)$$

$z := 13$. $\varepsilon := 1$.
Switch 13 and 14.

The bijection $\varphi : \mathcal{C}_{n+1} \to \mathcal{S}_n$. Fixing bad pairs

For each but the last cycle of σ, from left to right:

- $z :=$ rightmost entry of the cycle.
  If $\{z, z-1\}$ or $\{z, z+1\}$ are bad, let $\varepsilon = \pm 1$ be such that $\{z, z+\varepsilon\}$ is bad and $\sigma(z+\varepsilon)$ is largest.
- Repeat for as long as $\{z, z+\varepsilon\}$ is bad:
  1. Switch z and $z+\varepsilon$ (in the cycle form of σ).
  2. If the elements preceding the last switched entries have consecutive values, switch them. Repeat 2.
  3. $z :=$ new rightmost entry of the cycle.

$$\pi = (11, 4, 10, 1, 7, 16, 9, 3, 5, 12, 20, 2, 6, 14, 18, 8, 13, 19, 15, 17, 21)$$
$$\sigma = (11, 4, 9, 3, 5)(16, 10, 2, 6, 14)(20, 1, 7, 13, 18, 8, 12, 19, 15, 17)$$

$z := 13$. $\varepsilon := 1$.
Switch 13 and 14. Switch 6 and 7.

The bijection $\varphi : \mathcal{C}_{n+1} \to \mathcal{S}_n$. Fixing bad pairs

For each but the last cycle of σ, from left to right:

- $z :=$ rightmost entry of the cycle.
  If $\{z, z-1\}$ or $\{z, z+1\}$ are bad, let $\varepsilon = \pm 1$ be such that $\{z, z+\varepsilon\}$ is bad and $\sigma(z+\varepsilon)$ is largest.
- Repeat for as long as $\{z, z+\varepsilon\}$ is bad:
  1. Switch z and $z+\varepsilon$ (in the cycle form of σ).
  2. If the elements preceding the last switched entries have consecutive values, switch them. Repeat 2.
  3. $z :=$ new rightmost entry of the cycle.

$$\pi = (11, 4, 10, 1, 7, 16, 9, 3, 5, 12, 20, 2, 6, 14, 18, 8, 13, 19, 15, 17, 21)$$
$$\sigma = (11, 4, 9, 3, 5)(16, 10, 2, 7, 14)(20, 1, 6, 13, 18, 8, 12, 19, 15, 17)$$

$z := 13$. $\varepsilon := 1$.

Switch 13 and 14. Switch 6 and 7.

The bijection $\varphi : \mathcal{C}_{n+1} \to \mathcal{S}_n$. Fixing bad pairs

For each but the last cycle of σ, from left to right:

- $z :=$ rightmost entry of the cycle.
  If $\{z, z-1\}$ or $\{z, z+1\}$ are bad, let $\varepsilon = \pm 1$ be such that $\{z, z+\varepsilon\}$ is bad and $\sigma(z+\varepsilon)$ is largest.
- Repeat for as long as $\{z, z+\varepsilon\}$ is bad:
  1. Switch z and $z+\varepsilon$ (in the cycle form of σ).
  2. If the elements preceding the last switched entries have consecutive values, switch them. Repeat 2.
  3. $z :=$ new rightmost entry of the cycle.

$$\pi = (11, 4, 10, 1, 7, 16, 9, 3, 5, 12, 20, 2, 6, 14, 18, 8, 13, 19, 15, 17, 21)$$
$$\sigma = (11, 4, 9, 3, 5)(16, 10, 2, 7, 14)(20, 1, 6, 13, 18, 8, 12, 19, 15, 17)$$

$z := 13$. $\varepsilon := 1$.

Switch 13 and 14. Switch 6 and 7. Switch 2 and 1.

The bijection $\varphi : \mathcal{C}_{n+1} \to \mathcal{S}_n$. Fixing bad pairs

For each but the last cycle of σ, from left to right:

- $z :=$ rightmost entry of the cycle.
  If $\{z, z-1\}$ or $\{z, z+1\}$ are bad, let $\varepsilon = \pm 1$ be such that $\{z, z+\varepsilon\}$ is bad and $\sigma(z+\varepsilon)$ is largest.
- Repeat for as long as $\{z, z+\varepsilon\}$ is bad:
  1. Switch z and $z+\varepsilon$ (in the cycle form of σ).
  2. If the elements preceding the last switched entries have consecutive values, switch them. Repeat 2.
  3. $z :=$ new rightmost entry of the cycle.

$$\pi = (11, 4, 10, 1, 7, 16, 9, 3, 5, 12, 20, 2, 6, 14, 18, 8, 13, 19, 15, 17, 21)$$
$$\sigma = (11, 4, 9, 3, 5)(16, \underline{10}, 1, 7, 14)(\underline{20}, 2, 6, 13, 18, 8, 12, 19, 15, 17)$$

$z := 13$. $\varepsilon := 1$.

Switch 13 and 14. Switch 6 and 7. Switch 2 and 1.

The bijection $\varphi : \mathcal{C}_{n+1} \longrightarrow \mathcal{S}_n$. Fixing bad pairs

For each but the last cycle of σ, from left to right:

- $z :=$ rightmost entry of the cycle.
  If $\{z, z-1\}$ or $\{z, z+1\}$ are bad, let $\varepsilon = \pm 1$ be such that $\{z, z+\varepsilon\}$ is bad and $\sigma(z+\varepsilon)$ is largest.
- Repeat for as long as $\{z, z+\varepsilon\}$ is bad:
  1. Switch z and $z+\varepsilon$ (in the cycle form of σ).
  2. If the elements preceding the last switched entries have consecutive values, switch them. Repeat 2.
  3. $z :=$ new rightmost entry of the cycle.

$$\pi = (11, 4, 10, 1, 7, 16, 9, 3, 5, 12, 20, 2, 6, 14, 18, 8, 13, 19, 15, 17, 21)$$
$$\sigma = (11, 4, 9, 3, 5)(16, 10, 1, 7, 14)(20, 2, 6, 13, 18, 8, 12, 19, 15, 17)$$

$z := 14$. $\varepsilon := 1$.

The bijection $\varphi : \mathcal{C}_{n+1} \to \mathcal{S}_n$. Fixing bad pairs

For each but the last cycle of σ, from left to right:

- $z :=$ rightmost entry of the cycle.
  If $\{z, z-1\}$ or $\{z, z+1\}$ are bad, let $\varepsilon = \pm 1$ be such that $\{z, z+\varepsilon\}$ is bad and $\sigma(z+\varepsilon)$ is largest.
- Repeat for as long as $\{z, z+\varepsilon\}$ is bad:
  1. Switch z and $z+\varepsilon$ (in the cycle form of σ).
  2. If the elements preceding the last switched entries have consecutive values, switch them. Repeat 2.
  3. $z :=$ new rightmost entry of the cycle.

$$\pi = (11, 4, 10, 1, 7, 16, 9, 3, 5, 12, 20, 2, 6, 14, 18, 8, 13, 19, 15, 17, 21)$$
$$\sigma = (11, 4, 9, 3, 5)(16, 10, 1, 7, 14)(20, 2, 6, 13, 18, 8, 12, 19, 15, 17)$$

$z := 14$. $\{14, 15\}$ is bad. $\varepsilon := 1$.

The bijection $\varphi : \mathcal{C}_{n+1} \to \mathcal{S}_n$. Fixing bad pairs

For each but the last cycle of σ, from left to right:

- $z :=$ rightmost entry of the cycle.
  If $\{z, z-1\}$ or $\{z, z+1\}$ are bad, let $\varepsilon = \pm 1$ be such that $\{z, z+\varepsilon\}$ is bad and $\sigma(z+\varepsilon)$ is largest.
- Repeat for as long as $\{z, z+\varepsilon\}$ is bad:
  1. Switch z and $z+\varepsilon$ (in the cycle form of σ).
  2. If the elements preceding the last switched entries have consecutive values, switch them. Repeat 2.
  3. $z :=$ new rightmost entry of the cycle.

$$\pi = (11, 4, 10, 1, 7, 16, 9, 3, 5, 12, 20, 2, 6, 14, 18, 8, 13, 19, 15, 17, 21)$$
$$\sigma = (11, 4, 9, 3, 5)(16, 10, 1, 7, 14)(20, 2, 6, 13, 18, 8, 12, 19, 15, 17)$$

$z := 14$. $\varepsilon := 1$.
Switch 14 and 15.

The bijection $\varphi : \mathcal{C}_{n+1} \to \mathcal{S}_n$. Fixing bad pairs

For each but the last cycle of σ, from left to right:

- $z :=$ rightmost entry of the cycle.
  If $\{z, z-1\}$ or $\{z, z+1\}$ are bad, let $\varepsilon = \pm 1$ be such that $\{z, z+\varepsilon\}$ is bad and $\sigma(z+\varepsilon)$ is largest.
- Repeat for as long as $\{z, z+\varepsilon\}$ is bad:
  1. Switch z and $z+\varepsilon$ (in the cycle form of σ).
  2. If the elements preceding the last switched entries have consecutive values, switch them. Repeat 2.
  3. $z :=$ new rightmost entry of the cycle.

$$\pi = (11, 4, 10, 1, 7, 16, 9, 3, 5, 12, 20, 2, 6, 14, 18, 8, 13, 19, 15, 17, 21)$$
$$\sigma = (11, 4, 9, 3, 5)(16, 10, 1, 7, 15)(20, 2, 6, 13, 18, 8, 12, 19, 14, 17)$$

$z := 14$. $\varepsilon := 1$.
Switch 14 and 15.

The bijection $\varphi : \mathcal{C}_{n+1} \to \mathcal{S}_n$. Fixing bad pairs

For each but the last cycle of σ, from left to right:

- $z :=$ rightmost entry of the cycle.
  If $\{z, z-1\}$ or $\{z, z+1\}$ are bad, let $\varepsilon = \pm 1$ be such that $\{z, z+\varepsilon\}$ is bad and $\sigma(z+\varepsilon)$ is largest.
- Repeat for as long as $\{z, z+\varepsilon\}$ is bad:
  1. Switch z and $z+\varepsilon$ (in the cycle form of σ).
  2. If the elements preceding the last switched entries have consecutive values, switch them. Repeat 2.
  3. $z :=$ new rightmost entry of the cycle.

$$\pi = (11, 4, 10, 1, 7, 16, 9, 3, 5, 12, 20, 2, 6, 14, 18, 8, 13, 19, 15, 17, 21)$$
$$\sigma = (11, 4, 9, 3, 5)(16, 10, 1, \underline{7}, 15)(20, 2, 6, 13, 18, 8, 12, \underline{19}, 14, 17)$$

$z := 14$. $\varepsilon := 1$.

Switch 14 and 15.

The bijection $\varphi : \mathcal{C}_{n+1} \to \mathcal{S}_n$. Fixing bad pairs

For each but the last cycle of σ, from left to right:

- $z :=$ rightmost entry of the cycle.
  If $\{z, z-1\}$ or $\{z, z+1\}$ are bad, let $\varepsilon = \pm 1$ be such that $\{z, z+\varepsilon\}$ is bad and $\sigma(z+\varepsilon)$ is largest.
- Repeat for as long as $\{z, z+\varepsilon\}$ is bad:
  1. Switch z and $z+\varepsilon$ (in the cycle form of σ).
  2. If the elements preceding the last switched entries have consecutive values, switch them. Repeat 2.
  3. $z :=$ new rightmost entry of the cycle.

$$\pi = (11, 4, 10, 1, 7, 16, 9, 3, 5, 12, 20, 2, 6, 14, 18, 8, 13, 19, 15, 17, 21)$$
$$\sigma = (11, 4, 9, 3, 5)(16, 10, 1, 7, 15)(20, 2, 6, 13, 18, 8, 12, 19, 14, 17)$$

$z := 15.$ $\varepsilon := 1.$

The bijection $\varphi : \mathcal{C}_{n+1} \to \mathcal{S}_n$. Fixing bad pairs

For each but the last cycle of σ, from left to right:

- $z :=$ rightmost entry of the cycle.
  If $\{z, z-1\}$ or $\{z, z+1\}$ are bad, let $\varepsilon = \pm 1$ be such that $\{z, z+\varepsilon\}$ is bad and $\sigma(z+\varepsilon)$ is largest.
- Repeat for as long as $\{z, z+\varepsilon\}$ is bad:
  1. Switch z and $z+\varepsilon$ (in the cycle form of σ).
  2. If the elements preceding the last switched entries have consecutive values, switch them. Repeat 2.
  3. $z :=$ new rightmost entry of the cycle.

$$\pi = (11, 4, 10, 1, 7, 16, 9, 3, 5, 12, 20, 2, 6, 14, 18, 8, 13, 19, 15, 17, 21)$$
$$\sigma = (11, 4, 9, 3, 5)(16, 10, 1, 7, 15)(20, 2, 6, 13, 18, 8, 12, 19, 14, 17)$$

$z := 15$. $\{15, 16\}$ is OK, so we are done.

The bijection $\varphi : \mathcal{C}_{n+1} \longrightarrow \mathcal{S}_n$. Fixing bad pairs

For each but the last cycle of σ, from left to right:

- $z :=$ rightmost entry of the cycle.
  If $\{z, z-1\}$ or $\{z, z+1\}$ are bad, let $\varepsilon = \pm 1$ be such that $\{z, z+\varepsilon\}$ is bad and $\sigma(z+\varepsilon)$ is largest.
- Repeat for as long as $\{z, z+\varepsilon\}$ is bad:
  1. Switch z and $z+\varepsilon$ (in the cycle form of σ).
  2. If the elements preceding the last switched entries have consecutive values, switch them. Repeat 2.
  3. $z :=$ new rightmost entry of the cycle.

$$\pi = (11, 4, 10, 1, 7, 16, 9, 3, 5, 12, 20, 2, 6, 14, 18, 8, 13, 19, 15, 17, 21)$$
$$\varphi(\pi) = (11, 4, 9, 3, 5)(16, 10, 1, 7, 15)(20, 2, 6, 13, 18, 8, 12, 19, 14, 17)$$

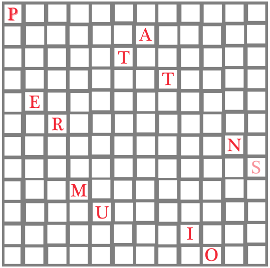
Define $\varphi(\pi) = \sigma$.

The descent sets are preserved

$$\pi = (11, 4, 10, 1, 7, 16, 9, 3, 5, 12, 20, 2, 6, 14, 18, 8, 13, 19, 15, 17, 21)$$
$$\varphi(\pi) = (11, 4, 9, 3, 5)(16, 10, 1, 7, 15)(20, 2, 6, 13, 18, 8, 12, 19, 14, 17)$$

In one-line notation,

$$\pi = 7 \cdot 6 \cdot 5\ 10\ 12\ 14\ 16 \cdot 13 \cdot 3 \cdot 1\ 4\ 20 \cdot 19 \cdot 18 \cdot 16 \cdot\ 9\ \ 21 \cdot 8\ 15 \cdot 2\ 11$$
$$\varphi(\pi) = 7 \cdot 6 \cdot 5\ \ 9\ \ 11\ 13\ 15 \cdot 12 \cdot 3 \cdot 1\ 4\ 19 \cdot 18 \cdot 17 \cdot 16 \cdot 10\ 20 \cdot 8\ 14 \cdot 2$$



Eighth International Conference on
Permutation Patterns, *PP 2010*

August 9-13, Dartmouth College, Hanover, NH

Invited speakers:

- Nik Ruškuc, University of St Andrews
- Richard Stanley, MIT

Deadline for submission of abstracts: June 1

Deadline for early registration: June 15

http://math.dartmouth.edu/~pp2010