Experimental Techniques Applied to Convergence of Rational Difference Equations

Emilie Hogan
eahogan@math.rutgers.edu
Rutgers University

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Joint with Doron Zeilberger

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Introduction

Goal: describe the limiting behavior of a sequence, $\{x_n\}_{n=-k}^{\infty}$, produced by a rational difference equation

$$x_{n+1} = R(x_n, x_{n-1}, \ldots, x_{n-k})$$

with

- arbitrary positive initial conditions, $x_{-k}, \ldots, x_0$, and
- R a rational function with numerator and denominator linear in $\{x_n, \ldots, x_{n-k}\}$, and all coefficients positive.

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For example

$$x_{n+1} = \frac{1}{\frac{9}{20} + x_n}$$

Convergence

If $\{x_n\}_{n=-k}^{\infty}$ is going to converge, it will be to an equilibrium point, $\bar{x}$, where $\bar{x}$ is a positive solution to

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There are two general types of convergence that people are interested in.

1. **(Local Asymptotic Stability)** Given any initial conditions in some region "close" to $\bar{x}$ we have $x_n \to \bar{x}$.
2. **(Global Asymptotic Stability)** Given any positive initial conditions we have $x_n \to \bar{x}$.

Method - Step 1 ("Move" the equilibrium from $\bar{x}$ to 0)

Let $z_n = x_n - \bar{x}$, and substitute into $x_{n+1} = R(x_n, x_{n-1}, \ldots, x_{n-k})$

$$z_{n+1} = R(z_n + \bar{x}, z_{n-1} + \bar{x}, \ldots, z_{n-k} + \bar{x}) - \bar{x}$$
$$z_{n+1} = R_0(z_n, z_{n-1}, \ldots, z_{n-k})$$

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$$x_{n+1} = \frac{1}{\frac{9}{20} + x_n}$$
$$z_{n+1} = \frac{1}{\frac{9}{20} + \left(z_n + \frac{4}{5}\right)} - \frac{4}{5}$$
$$z_{n+1} = -\frac{16}{5} \cdot \frac{z_n}{5 + 4z_n}$$

Method - Step 2 (“Moving window” map)

Consider the map $Q : \mathbb{R}^{k+1} \to \mathbb{R}^{k+1}$, where

$$Q(\langle z_{n-k}, z_{n-k+1}, \ldots, z_n \rangle) = \langle z_{n-k+1}, \ldots, z_n, R_0(z_n, \ldots, z_{n-k}) \rangle$$

Think of this as a “moving window” on the sequence $\{z_n\}_{n=-k}^{\infty}$. Let

$$\mathcal{Z}_n := \langle z_{n-k}, \ldots, z_n \rangle$$

then $\mathcal{Z}_n = Q^n(\mathcal{Z}_0)$ where $\mathcal{Z}_0 = \langle z_{-k}, \ldots, z_0 \rangle$ is the vector of initial conditions.

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Global asymptotic stability (GAS) can now be stated as:

$$\lim_{n\to\infty} \mathcal{Z}_n = \lim_{n\to\infty} Q^n(\mathcal{Z}_0) = \langle 0, \ldots, 0 \rangle =: \bar{0}$$

for any admissible initial conditions $\mathcal{Z}_0$.

Method - Step 3 (Find a K)

Claim

If there exists a $K \in \mathbb{Z}_{>0}$ and a $0 < \delta < 1$ such that

$$\frac{|Q^K(\langle z_1, \ldots, z_{k+1}\rangle)|}{|\langle z_1, \ldots, z_{k+1}\rangle|} < \delta$$

for any admissible $\langle z_1, \ldots, z_{k+1}\rangle$ then

$$\lim_{n\to\infty} Q^n(\mathcal{Z}_0) = \bar{0}$$

as long as the initial conditions, $\mathcal{Z}_0$, are admissible.

Note, $|\cdot|$ is the Euclidean norm.

Why this works $\left(\frac{|Q^K(\langle z_1,\ldots,z_{k+1}\rangle)|}{|\langle z_1,\ldots,z_{k+1}\rangle|} < \delta\right)$

Consider the first K iterations of Q

$$\mathcal{Z}_0 = \langle z_{-k}, \ldots, z_0 \rangle$$
$$\vdots$$
$$\mathcal{Z}_{K-1} = \left\langle z_{-k+(K-1)}, \ldots, z_{K-1} \right\rangle$$

Let $\mathcal{Z} := \max_{0 \le i \le K-1} |\mathcal{Z}_i|$.

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Let $\mathcal{Z} := \max_{0 \le i \le K-1} |\mathcal{Z}_i|$. If the conditions of the claim are satisfied:

$$|(Q^K)^N(\mathcal{Z}_i)| < \delta^N |\mathcal{Z}_i| \le \delta^N \mathcal{Z}$$

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Let $\mathcal{Z} := \max_{0\le i\le K-1} |\mathcal{Z}_i|$. If the conditions of the claim are satisfied:

$$|Q^{NK+i}(\mathcal{Z}_0)| = |(Q^K)^N(\mathcal{Z}_i)| < \delta^N |\mathcal{Z}_i| \le \delta^N \mathcal{Z}$$

and since $\delta < 1$ the RHS goes to 0 as N goes to ∞.

Example

Consider our running example,

$$x_{n+1} = \frac{1}{\frac{9}{20} + x_n} \longleftrightarrow z_{n+1} = -\frac{16}{5} \cdot \frac{z_n}{5 + 4z_n}$$

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where $\bar{x} = \frac{4}{5}$. I claim that $K = 2$ will work. If

$$\max_{z_1 > -4/5} \frac{|Q^2(z_1)|}{|z_1|} = \delta < 1$$

then we're done.

Example (cont.)

$$Q^2(z_1) = \frac{256}{5} \cdot \frac{z_1}{125 + 36z_1}$$

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Q^2(z_1) &= \frac{256}{5} \cdot \frac{z_1}{125 + 36z_1} \\
\frac{|Q^2(z_1)|}{|z_1|} &= \frac{\frac{256}{5} \cdot \frac{|z_1|}{|125+36z_1|}}{|z_1|} \\
&= \frac{256}{5} \cdot \frac{1}{|125 + 36z_1|} \\
&<^* \frac{256}{5} \cdot \frac{1}{481/5} \\
&= \frac{256}{481} < 1
\end{aligned}
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* If $z_1 > -4/5$.

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&= \frac{256}{481} < 1
\end{aligned}$$

* If $z_1 > -4/5$.

Therefore, the rational difference equation $x_{n+1} = \frac{1}{9/20+x_n}$ is *globally asymptotically stable.*

Remarks

- Most questions in this subject deal with, e.g.,

$$x_{n+1} = \frac{1}{A + x_n}, \text{ or } x_{n+1} = \frac{\alpha + x_n}{A + x_{n-1}}$$

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  finding values for the parameters that guarantee types of convergence
- Currently our approach only deals with the case when the parameters assume specific values.
- Our approach boils down to showing that the maximum of some rational function is less than 1.
- Using Maple to conjecture these K values.

References

M.R.S. Kulenovic, G. Ladas, Dynamics of Second Order Rational Difference Equations, *Chapman and Hall/CRC press*, 2001.

E. Camouzis, G. Ladas, Dynamics of Third Order Rational Difference Equations, *Chapman and Hall/CRC press*, 2008.

Happy Birthday Dr. Z!