TANGENT AND SECANT q-CALCULUS: AND (t, q)-CALCULUS

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(Strasbourg)

Z = 60 Doron's Celebration
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Based on the paper

"The (t, q)-analogs of secant and tangent numbers"

jointly written with Guo-Niu Han

(t,q)-analog: the order matters!

For q,t-analogs and (q,t)-analogs see

Haiman-Woo (2007)

Reiner-Stanton (2009), respectively.

SUMMARY

One of the Gian-Carlo Rota legacies

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Standard q-notations

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The trick of the graded form

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ONE OF THE GIAN-CARLO ROTA LEGACIES

Every mathematician has only a few tricks in his pocket.

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A single one in this lecture.

STANDARD q-NOTATIONS

The q-ascending factorial: $(\omega; q)_0 := 1$ and for $k \geq 1$

$$(\omega; q)_k := (1 - \omega)(1 - \omega q) \cdots (1 - \omega q^{k-1});$$

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the q-binomial coefficient

$$\begin{bmatrix} n \\ k \end{bmatrix}_q := \frac{(q;q)_n}{(q;q)_k\,(q;q)_{n-k}} \qquad (0 \leq k \leq n);$$

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the q-analogs of the integers and factorials

$$[n]_q := \frac{1-q^n}{1-q} = 1 + q + q^2 + \cdots + q^{n-1};$$

$$[n]!_q := \frac{(q;q)_n}{(1-q)^n} = [n]_q[n-1]_q\cdots[1]_q.$$

THE TRICK OF THE GRADED FORM

Consider:

$$G(1,u) \\ = \sum_n A_n(1,1)\, u^n/n!$$

$G(1,u)$ the exp. g.f. for the sequence $(A_n(1,1))$;

THE TRICK OF THE GRADED FORM

Consider:

$$G(1,u) = \sum_n A_n(1,1)\,\frac{u^n}{n!} \qquad\qquad G(q;u) = \sum_n A_n(1,q)\,\frac{u^n}{(q;q)_n}$$

$G(1,u)$ the exp. g.f. for the sequence $(A_n(1,1))$;

$G(q;u)$ the q-factorial g.f. for the sequence $(A_n(1,q))$;

THE TRICK OF THE GRADED FORM

The horizontal arrow makes sense

$$G(1,u) = \sum_n A_n(1,1)\,\frac{u^n}{n!} \longrightarrow G(q;u) = \sum_n A_n(1,q)\,\frac{u^n}{(q;q)_n}$$

if

(1) $$\lim_{q\to 1} G(q;u(1-q)) = G(u).$$

THE TRICK OF THE GRADED FORM

Now, let $(G_r(q;u))_{(r\geq 0)}$ be a sequence of q-series such that

$$(2) \qquad \lim_r G_r(q;u) = G(q;u).$$

Form the graded form $\sum_{r\geq 0} t^r G_r(q;u)$.

THE TRICK OF THE GRADED FORM

Now, let $(G_r(q;u))_{(r\geq 0)}$ be a sequence of q-series such that

(2) $$\lim_r G_r(q;u) = G(q;u).$$

Form the graded form $\sum_{r\geq 0} t^r G_r(q;u)$.

Then

(3) $$(1-t)\sum_{r\geq 0} t^r G_r(q;u)\,\Big|_{t\,=\,1} = G(q;u).$$

THE TRICK OF THE GRADED FORM

The relation

$$(3) \qquad (1-t)\sum_{r\ge 0} t^r G_r(q;u)\,\Big|_{t\,=\,1} = G(q;u).$$

gives a sense to the vertical arrow

$$\sum_r t^r G_r(q;u) = \sum_n A_n(t,q)\,\frac{u^n}{(t;q)_{n+1}}$$

$$\uparrow$$

$$G(1,u) = \sum_n A_n(1,1)\,\frac{u^n}{n!} \longrightarrow G(q;u) = \sum_n A_n(1,q)\,\frac{u^n}{(q;q)_n}$$

(t,q)-ANALOGS OF SECANT AND TANGENT

Apply this trick of the graded forms to the sequences of

the secant numbers (E_{2n}) $(n \geq 0)$

and the tangent numbers (T_{2n+1}) $(n \geq 0)$.

(t, q)-ANALOGS OF SECANT AND TANGENT

The secant numbers E_{2n} $(n \geq 0)$ defined by

$$\sec u = \frac{1}{\cos u} = 1 + \sum_{n \geq 1} \frac{u^{2n}}{(2n)!} E_n$$

$$= 1 + \frac{u^2}{2!} 1 + \frac{u^4}{4!} 5 + \frac{u^6}{6!} 61 + \frac{u^8}{8!} 1385 + \frac{u^{10}}{10!} 50521 + \cdots$$

Sloane's Encyclopedia A122045.

(t,q)-ANALOGS OF SECANT AND TANGENT

The tangent numbers T_{2n+1} $(n \geq 0)$ by

$$\tan u = \sum_{n\geq 0} \frac{u^{2n+1}}{(2n+1)!} T_{2n+1}$$
$$= \frac{u}{1!}1 + \frac{u^3}{3!}2 + \frac{u^5}{5!}16 + \frac{u^7}{7!}272 + \frac{u^9}{9!}7936 + \frac{u^{11}}{11!}353792 + \cdots$$

Sloane's Encyclopedia A000182

(t, q)-ANALOGS OF SECANT AND TANGENT

Huge formulary, see:

Niels Nielsen, Traité élémentaire des nombres de Bernoulli, Paris, Gauthier-Villars 1923.

(t,q)-ANALOGS OF SECANT AND TANGENT

Work out a (t,q)-analog with

$$G(u) = \sec u \quad \text{or} \quad \tan u.$$

(t, q)-ANALOGS OF SECANT AND TANGENT

Jackson (1904) introduced both

$$\sin_q(u) := \sum_{n\geq 0} (-1)^n \frac{u^{2n+1}}{(q;q)_{2n+1}};$$

$$\cos_q(u) := \sum_{n\geq 0} (-1)^n \frac{u^{2n}}{(q;q)_{2n}};$$

(t,q)-ANALOGS OF SECANT AND TANGENT

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$$\cos_q(u) := \sum_{n\geq 0} (-1)^n \frac{u^{2n}}{(q;q)_{2n}};$$

so that *q-tangent* and *q-secant* are defined by:

$$\tan_q(u) := \frac{\sin_q(u)}{\cos_q(u)} = \sum_{n\geq 0} \frac{u^{2n+1}}{(q;q)_{2n+1}} T_{2n+1}(q);$$

$$\sec_q(u) := \frac{1}{\cos_q(u)} = \sum_{n\geq 0} \frac{u^{2n}}{(q;q)_{2n}} E_{2n}(q).$$

(t, q)-ANALOGS OF SECANT AND TANGENT

Have $T_{2n+1}(q)$ and $E_{2n}(q)$ been studied?

Not so much,

only in scattered papers (Stanley (1976), Andrews-Gessel (1978), ..., and a few others)

or Oberwolfach talks (Schützenberger (1975)).

(t,q)-ANALOGS OF SECANT AND TANGENT

First values:

$$E_0(q) = E_2(q) = 1, \quad E_4(q) = q(1+q)^2 + q^4,$$

$$E_6(q) = q^2(1+q)^2(1+q^2+q^4)(1+q+q^2+2q^3) + q^{12},$$

$$T_1(q) = 1, \quad T_3(q) = q(1+q),$$

$$T_5(q) = q^2(1+q)^2(1+q^2)^2,$$

$$T_7(q) = q^3(1+q)^2(1+q^2)(1+q^3)(1+q+3q^2+2q^3+3q^4+ 2q^5+3q^6+q^7+q^8).$$

(t,q)-ANALOGS OF SECANT AND TANGENT

Appropriate start:

$$\sec u \longrightarrow \sec_q(u)$$

$$= \sum_n E_{2n} \frac{u^{2n}}{(2n)!} \qquad = \sum_n E_{2n}(q) \frac{u^{2n}}{(q;q)_{2n}}$$

$$\tan u \longrightarrow \tan_q(u)$$

$$= \sum_n T_{2n+1} \frac{u^{2n+1}}{(2n+1)!} \qquad = \sum_n T_{2n+1}(q) \frac{u^n}{(q;q)_{2n+1}}$$

as

$$\lim_{q\to 1} \sec_q(u(1-q)) = \sec u;$$

$$\lim_{q\to 1} \tan_q(u(1-q)) = \tan u.$$

(t,q)-ANALOGS OF SECANT AND TANGENT

Find out two sequences $(\sec_q^{(r)}(u))$ $(\tan_q^{(r)}(u))$ $(r \ge 0)$

such that

$$\lim_r \sec_q^{(r)}(u) = \sec_q(u)$$

and $$\lim_r \tan_q^{(r)}(u) = \tan_q(u).$$

(t,q)-ANALOGS OF SECANT AND TANGENT

Take

$$\sin_q^{(r)}(u) := \sum_{n\geq 0}(-1)^n \frac{(q^r;q)_{2n+1}}{(q;q)_{2n+1}} u^{2n+1};$$

$$\cos_q^{(r)}(u) := \sum_{n\geq 0}(-1)^n \frac{(q^r;q)_{2n}}{(q;q)_{2n}} u^{2n};$$

(t, q)-ANALOGS OF SECANT AND TANGENT

Take

$$\sin_q^{(r)}(u) := \sum_{n\geq 0}(-1)^n \frac{(q^r;q)_{2n+1}}{(q;q)_{2n+1}} u^{2n+1};$$

$$\cos_q^{(r)}(u) := \sum_{n\geq 0}(-1)^n \frac{(q^r;q)_{2n}}{(q;q)_{2n}} u^{2n};$$

$$\sec_q^{(r)}(u) := \frac{1}{\cos_q^{(r)}(u)};$$

$$\tan_q^{(r)}(u) := \frac{\sin_q^{(r)}(u)}{\cos_q^{(r)}(u)};$$

(t,q)-ANALOGS OF SECANT AND TANGENT

and verify

$\lim_r \sec_q^{(r)}(u) = \sec_q(u)$ and $\lim_r \tan_q^{(r)}(u) = \tan_q(u)$.

(t,q)-ANALOGS OF SECANT AND TANGENT

Then, define $E_{2n}(t,q)$ by

$$\sec u = \sum_n E_{2n} \frac{u^{2n}}{(2n)!} \longrightarrow \sec_q(u) = \sum_n E_{2n}(q) \frac{u^n}{(q;q)_{2n}}$$

$$\longrightarrow \sum_r t^r \sec_q^{(r)}(u) = \sum_n E_{2n}(t,q) \frac{u^{2n}}{(t;q)_{2n+1}}$$

(t, q)-ANALOGS OF SECANT AND TANGENT

And define $T_{2n+1}(t, q)$ by

$$\sum_r t^r \tan_q^{(r)}(u) = \sum_n T_{2n+1}(t, q) \frac{u^{2n+1}}{(t; q)_{2n+2}}$$

$$\uparrow$$

$$\tan u \longrightarrow \tan_q(u)$$

$$= \sum_n T_{2n+1} \frac{u^{2n+1}}{(2n+1)!} \qquad = \sum_n T_{2n+1}(q) \frac{u^n}{(q; q)_{2n+1}}$$

(t, q)-ANALOGS OF SECANT AND TANGENT

First values:

$E_0(t, q) = 1$; $E_2(t, q) = t$; $E_4(t, q) = t^2 q(1 + 2q + q^2 + tq^3)$;

$$E_6(t, q) = t^2 q^2 (1 + 2q + q^2 + tq(1 + 4q + 8q^2 + 10q^3 + 8q^4 + 4q^5 + q^6) + t^2 q^5 (2 + 5q + 6q^2 + 5q^3 + 2q^4) + t^3 q^{10});$$

(t, q)-ANALOGS OF SECANT AND TANGENT

First values:

$T_1(t, q) = t$; $T_3(t, q) = t^2 q(1 + q)$;

$T_5(t, q) = t^2 q^2 (1 + q)(1 + tq(1 + 2q + 2q^2 + q^3) + t^2 q^6)$;

$$T_7(t, q) = t^2 q^3 (1 + q)(1 + tq(2 + 5q + 7q^2 + 7q^3 + 5q^4 + 2q^5) + t^2 q^3 (1 + 4q + 10q^2 + 15q^3 + 18q^4 + 15q^5 + 10q^6 + 4q^7 + q^8) + t^3 q^8 (2 + 5q + 7q^2 + 7q^3 + 5q^4 + 2q^5) + t^4 q^{14}).$$

(t, q)-ANALOGS OF SECANT AND TANGENT

Recall: $E_{2n}(t, q)$ defined by

$$\sum_{r\geq 0} t^r \frac{1}{\sum_{n\geq 0} (-1)^n \frac{(q^r; q)_{2n}}{(q; q)_{2n}} u^{2n}} = \sum_{n\geq 0} E_{2n}(t, q) \frac{u^{2n}}{(t; q)_{2n+1}}.$$

Prove that each $E_{2n}(t, q)$ is a polynomial with positive integral coefficients.

(t,q)-ANALOGS OF SECANT AND TANGENT

Also $T_{2n+1}(t,q)$ defined by

$$\sum_{r\geq 0} t^r \frac{\sum_{n\geq 0}(-1)^n \frac{(q^r;q)_{2n+1}}{(q;q)_{2n+1}}u^{2n+1}}{\sum_{n\geq 0}(-1)^n \frac{(q^r;q)_{2n}}{(q;q)_{2n}}u^{2n}} = \sum_{n\geq 0} T_{2n+1}(t,q)\frac{u^{2n+1}}{(t;q)_{2n+2}}.$$

Prove that each $T_{2n+1}(t,q)$ is a polynomial with positive integral coefficients.

(t,q)-ANALOGS OF SECANT AND TANGENT

For

$$\sec_q(u) = \sum_{n\geq 0} E_{2n}(q)\frac{u^{2n}}{(q;q)_{2n}}$$

and

$$\tan_q(u) = \sum_{n\geq 0} T_{2n+1}(q)\frac{u^{2n+1}}{(q;q)_{2n+1}}$$

easy: $E_{2n}(q)$ and $T_{2n+1}(q)$ are polynomials with positive integral coefficients:

(t, q)-ANALOGS OF SECANT AND TANGENT

For
$$\sec_q(u) = \sum_{n\ge 0} E_{2n}(q)\frac{u^{2n}}{(q;q)_{2n}}$$
and
$$\tan_q(u) = \sum_{n\ge 0} T_{2n+1}(q)\frac{u^{2n+1}}{(q;q)_{2n+1}}$$
easy: $E_{2n}(q)$ and $T_{2n+1}(q)$ are polynomials with positive integral coefficients:

Just q-mimick the differential properties of secant and tangent,

using the q-binomial theorem.

(t,q)-ANALOGS OF SECANT AND TANGENT

For $E_{2n}(t,q)$ and $T_{2n+1}(t,q)$ only use the very definitions:

$$\sum_{r\geq 0} t^r \frac{1}{\displaystyle\sum_{n\geq 0}(-1)^n \frac{(q^r;q)_{2n}}{(q;q)_{2n}} u^{2n}} = \sum_{n\geq 0} E_{2n}(t,q) \frac{u^{2n}}{(t;q)_{2n+1}};$$

$$\sum_{r\geq 0} t^r \frac{\displaystyle\sum_{n\geq 0}(-1)^n \frac{(q^r;q)_{2n+1}}{(q;q)_{2n+1}} u^{2n+1}}{\displaystyle\sum_{n\geq 0}(-1)^n \frac{(q^r;q)_{2n}}{(q;q)_{2n}} u^{2n}} = \sum_{n\geq 0} T_{2n+1}(t,q) \frac{u^{2n+1}}{(t;q)_{2n+2}}.$$

(t,q)-ANALOGS OF SECANT AND TANGENT

For $E_{2n}(t,q)$ and $T_{2n+1}(t,q)$ only use the very definitions:

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$$\sum_{r\geq 0} t^r \frac{\sum_{n\geq 0}(-1)^n \frac{(q^r;q)_{2n+1}}{(q;q)_{2n+1}} u^{2n+1}}{\sum_{n\geq 0}(-1)^n \frac{(q^r;q)_{2n}}{(q;q)_{2n}} u^{2n}} = \sum_{n\geq 0} T_{2n+1}(t,q) \frac{u^{2n+1}}{(t;q)_{2n+2}}.$$

Recourse to combinatorial methods.

Refer to the objects counted by secant and tangent.

(t, q)-ANALOGS OF SECANT AND TANGENT

Following Désiré André (1881) each permutation

$$\sigma = \sigma(1) \cdots \sigma(n)$$

is said to be alternating if $\sigma(1) < \sigma(2)$, $\sigma(2) > \sigma(3)$, $\sigma(3) < \sigma(4)$, etc. in an alternating way.

Let $\mathcal{T}_n$ designate the set of alternating permutations of order n. Désiré André showed that

$$\#\mathcal{T}_{2n+1} = T_{2n+1}, \qquad \#\mathcal{T}_{2n} = E_{2n}.$$

(t,q)-ANALOGS OF SECANT AND TANGENT

With "inv" being the number of inversions

q-mimick Désiré André's derivation:

$$\sum_{\sigma\in\mathcal{T}_n} q^{\operatorname{inv}\sigma} = E_n(q) = E_n(1,q)$$

if n even and $=T_n$ if n odd.

(t,q)-ANALOGS OF SECANT AND TANGENT

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if n even and $= T_n$ if n odd.

But what to do with the variable "t"?

Look for other statistics.

(t, q)-ANALOGS OF SECANT AND TANGENT

For each permutation $\sigma = \sigma(1) \cdots \sigma(n)$ let

$$\mathrm{IDES}\,\sigma := \{\sigma(i) : 1 + \sigma(i) = \sigma(j) \text{ for } 1 \le j \le i-1\};$$

(t, q)-ANALOGS OF SECANT AND TANGENT

For each permutation $\sigma = \sigma(1) \cdots \sigma(n)$ let

$$\begin{aligned} \mathrm{IDES}\,\sigma &:= \{\sigma(i) : 1 + \sigma(i) = \sigma(j) \text{ for } 1 \le j \le i - 1\}; \\ \mathrm{ides}\,\sigma &:= \#\,\mathrm{IDES}\,\sigma; \end{aligned}$$

(t,q)-ANALOGS OF SECANT AND TANGENT

For each permutation $\sigma = \sigma(1)\cdots\sigma(n)$ let

$$\begin{aligned}
\mathrm{IDES}\,\sigma &:= \{\sigma(i) : 1+\sigma(i)=\sigma(j) \text{ for } 1\le j\le i-1\};\\
\mathrm{ides}\,\sigma &:= \#\,\mathrm{IDES}\,\sigma;\\
\mathrm{imaj}\,\sigma &:= \sum_{\sigma(i)\in\mathrm{IDES}\sigma} \sigma(i).
\end{aligned}$$

(t,q)-ANALOGS OF SECANT AND TANGENT

For each permutation $\sigma = \sigma(1)\cdots\sigma(n)$ let

$$\begin{aligned}
\mathrm{IDES}\,\sigma &:= \{\sigma(i) : 1+\sigma(i) = \sigma(j) \text{ for } 1 \le j \le i-1\};\\
\mathrm{ides}\,\sigma &:= \#\,\mathrm{IDES}\,\sigma;\\
\mathrm{imaj}\,\sigma &:= \sum_{\sigma(i)\in\mathrm{IDES}\sigma} \sigma(i).
\end{aligned}$$

$$\sigma = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 \\ 2 & 5 & 1 & 9 & 6 & 7 & 4 & 8 & 3 \end{pmatrix}$$

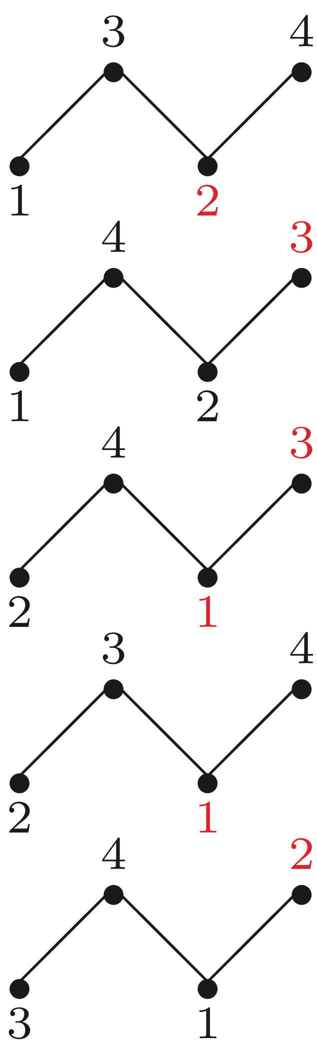
$\mathrm{IDES}\,\sigma = \{1,3,4,8\}$; $\quad \mathrm{ides}\,\sigma = 4$; $\quad \mathrm{imaj}\,\sigma = 16$.

(t, q)-ANALOGS OF SECANT AND TANGENT

Try "imaj" as we know that

$$\sum_{\sigma \in \mathcal{T}_n} q^{\operatorname{inv} \sigma} = \sum_{\sigma \in \mathcal{T}_n} q^{\operatorname{imaj} \sigma}.$$

(t,q)-ANALOGS OF SECANT AND TANGENT



inv	imaj
1	2
2	3
3	4
2	1
4	2

$$E_4(q) = q(1 + 2q + q^2 + q^3)$$

(t,q)-ANALOGS OF SECANT AND TANGENT

The most "natural" statistic that can be associated with

"imaj"

is

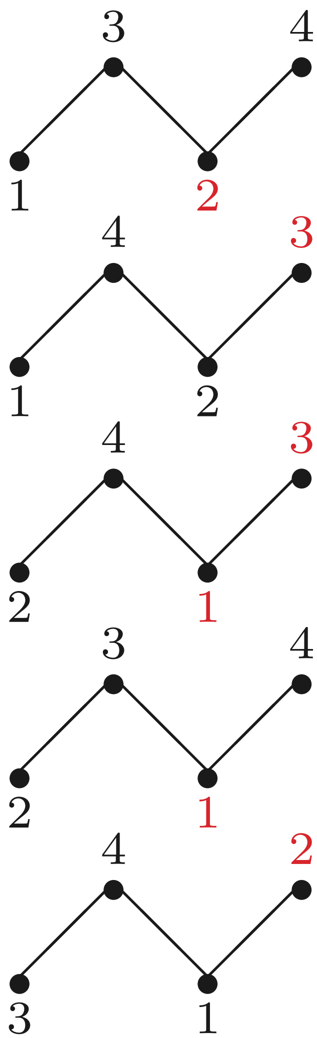
"ides."

Not quite, but

"1+ides"

will do the job!

(t,q)-ANALOGS OF SECANT AND TANGENT



1+ides	imaj	$t^{1+\text{ides}}q^{\text{imaj}}$
2	2	t^2q^2
2	3	t^2q^3
3	4	t^3q^4
2	1	t^2q
2	2	t^2q^2

$$E_4(t,q) = t^2q(1 + 2q + q^2 + tq^3)$$

(t, q)-ANALOGS OF SECANT AND TANGENT

Theorem. *The coefficients of the graded forms of* $\tan_q(u)$ *and* $\sec_q(u)$ *are generating polynomials for the sets of alternating permutations by the pair* $(1 + \operatorname{ides}, \operatorname{imaj})$*:*

$$T_{2n+1}(t, q) = \sum_{\sigma \in \mathcal{T}_{2n+1}} t^{1+\operatorname{ides}\sigma} q^{\operatorname{imaj}\sigma};$$

$$E_{2n}(t, q) = \sum_{\sigma \in \mathcal{T}_{2n}} t^{1+\operatorname{ides}\sigma} q^{\operatorname{imaj}\sigma}.$$

In particular, $T_{2n+1}(t, q)$ *and* $E_{2n}(t, q)$ *are*

polynomials with positive integral coefficients.

(t,q)-ANALOGS OF SECANT AND TANGENT

Is there a computer proof?

(t, q)-ANALOGS OF SECANT AND TANGENT

Is there a computer proof?

Even a computer-aided proof?

(t, q)-ANALOGS OF SECANT AND TANGENT

Is there a computer proof?

Even a computer-aided proof?

Soon, wait for the pair: DORON - ELKHAD,

we do celebrate.