The Joys of Mathematics with Doron

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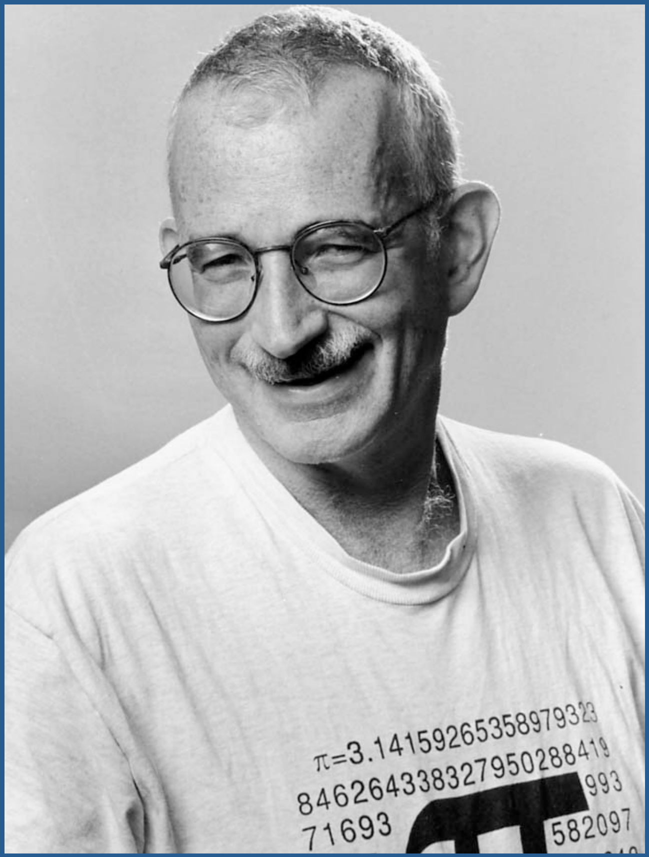
MAA



From A=B to Z=60
DIMACS
May 28, 2010

PowerPoint available at
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מכון ויצמן למדע
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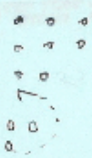
March 12, 198[illegible]

Dear David,

Thanks a lot for your reprints. I was particularly impressed and very much enjoyed the easy proof of the R-R identities. Did you move to Wisconsin or are you going back to Penn state? Here is a direct combinatorial proof of Corollary 1.8 (p.12) in Andrews' book on partition:

$$p(n) + p(n-5) + p(n-7) \ldots + p(n-m(6m-1)) + p(n-m(6m+1)) + \ldots = p(n-1) + p(n-2) + \ldots + p(n-\tfrac{1}{2}(2m+1)(6m+2)) + p(n-\tfrac{1}{2}(2m+1)(6m+4))$$

the l.h.s. counts pairs of partitions (π, σ) where π is a regular partition and σ is a certain pentagonal partition, $|\pi| + |\sigma| = n$, e.g. if $n = 11$



if the smallest part of the white partition is $\leq$ smallest part of black partition move it to the black par getting

Else convert the smallest black part into white

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Euler's Theorem:

$$\prod_{n=1}^{\infty}\left(1-q^{n}\right)=\sum_{k=-\infty}^{\infty}(-1)^{k}\,q^{\frac{3}{2}k^{2}+\frac{1}{2}k}$$

Combinatorial interpretation:

# of partitions of n into even # distinct parts minus # of partitions of n into odd # distinct parts	=	$(-1)^k$, if n is a pentagonal number ($n = {}^{3}/_{2}\,k^{2} + \frac{1}{2}\,k$) 0, if not

There is a simple, bijective proof of this due to Fabian Franklin.

Equivalent Theorem:

$$1=\frac{1}{\prod_{n=1}^{\infty}\left(1-q^{n}\right)}\sum_{k=-\infty}^{\infty}(-1)^{k}\,q^{\frac{3}{2}k^{2}+\frac{1}{2}k}$$

Combinatorial interpretation: For $n \geq 1$ and k in $\boldsymbol{Z}$,

$$\sum_{k\text{ even}} p\left(n-\tfrac{3}{2}k^{2}-\tfrac{1}{2}k\right)=\sum_{k\text{ odd}} p\left(n-\tfrac{3}{2}k^{2}-\tfrac{1}{2}k\right)$$

Garsia and Milne had used their involution principle to turn Franklin’s proof into a bijection between the two sets represented by the sides of this equality.

Doron’s question: Is there a more natural bijection?

$$\sum p\left(n - \tfrac{3}{2}k^2 - \tfrac{1}{2}k\right)$$

This sum counts pairs (π , k), where π is a partition of $n - \tfrac{3}{2}k^2 - \tfrac{1}{2}k$ and k can be any integer; positive, negative, or zero.

B-Z: Bijecting Euler's Partitions-Recurrence, *American Math Monthly*, 1985:

t = # of parts in π, l = largest part in π

If $l - t \leq 3k$, subtract 1 from each part, construct new largest part of size $t + 3k - 1$, k becomes $k - 1$.

If $l - t > 3k$, remove largest part, add 1 to $l - 3k - 2$ parts, some of which may be 0, k becomes $k + 1$.

Doron's next challenge:

Let's produce a bijective proof of the Rogers-Ramanujan identities.

Garsia and Milne, Method for constructing bijections for classical partition identities, *Proc. Nat. Acad. Sci. USA*, 1981

———, A Rogers-Ramanujan Bijection, *J. Combin. Th. A*, 1981.

B., Easy Proof of the Rogers-Ramanujan Identities, *J. Number Theory*, 1983

$$(-xq;q)_s\left(-x^{-1};q\right)_s=\sum_m x^m q^{(m^2+m)/2}\begin{bmatrix}2s\\ s-m\end{bmatrix}$$

$$\frac{1}{(q;q)_{n+j}}=\sum_{i\ge 0}\frac{q^{i^2+2ij}}{(q;q)_{i+2j}}\begin{bmatrix}n-j\\ i\end{bmatrix}$$

$$\Rightarrow\quad \sum_j \frac{x^j q^{aj^2}}{(q;q)_{n-j}(q;q)_{n+j}}=\sum_{s\ge 0}\frac{q^{s^2}}{(q;q)_{n-s}}\sum_j \frac{x^j q^{(a-1)j^2}}{(q;q)_{s-j}(q;q)_{s+j}},$$

$$\frac{1}{(q;q)_\infty}\sum_j x^j q^{aj^2}=\sum_{s\ge 0} q^{s^2}\sum_j \frac{x^j q^{(a-1)j^2}}{(q;q)_{s-j}(q;q)_{s+j}}$$

$$\frac{\left(-xq^{3};q^{5}\right)_{\infty}\left(-x^{-1}q^{2};q^{5}\right)_{\infty}\left(q^{5};q^{5}\right)_{\infty}}{\left(q;q\right)_{\infty}}=\frac{1}{\left(q;q\right)_{\infty}}\sum_{j}x^{j}q^{\left(5j^{2}+j\right)/2}$$

$$\frac{\left(-xq^{3};q^{5}\right)_{\infty}\left(-x^{-1}q^{2};q^{5}\right)_{\infty}\left(q^{5};q^{5}\right)_{\infty}}{\left(q;q\right)_{\infty}}=\frac{1}{\left(q;q\right)_{\infty}}\sum_{j}x^{j}q^{\left(5j^{2}+j\right)/2}$$

$$=\sum_{s\geq 0}q^{s^{2}}\sum_{j}\frac{x^{j}q^{\left(3j^{2}+j\right)/2}}{\left(q;q\right)_{s-j}\left(q;q\right)_{s+j}}$$

$$\frac{\left(-xq^{3};q^{5}\right)_{\infty}\left(-x^{-1}q^{2};q^{5}\right)_{\infty}\left(q^{5};q^{5}\right)_{\infty}}{(q;q)_{\infty}}=\frac{1}{(q;q)_{\infty}}\sum_{j}x^{j}q^{\left(5j^{2}+j\right)/2}$$

$$=\sum_{s\geq 0}q^{s^{2}}\sum_{j}\frac{x^{j}q^{\left(3j^{2}+j\right)/2}}{(q;q)_{s-j}(q;q)_{s+j}}$$

$$=\sum_{s,t\geq 0}\frac{q^{s^{2}+t^{2}}}{(q;q)_{s-t}}\sum_{j}\frac{x^{j}q^{\left(j^{2}+j\right)/2}}{(q;q)_{t-j}(q;q)_{t+j}}$$

$$\frac{\left(-xq^3;q^5\right)_\infty \left(-x^{-1}q^2;q^5\right)_\infty \left(q^5;q^5\right)_\infty}{(q;q)_\infty} = \frac{1}{(q;q)_\infty}\sum_j x^j q^{\left(5j^2+j\right)/2}$$

$$= \sum_{s\geq 0} q^{s^2} \sum_j \frac{x^j q^{\left(3j^2+j\right)/2}}{(q;q)_{s-j}(q;q)_{s+j}}$$

$$= \sum_{s,t\geq 0} \frac{q^{s^2+t^2}}{(q;q)_{s-t}} \sum_j \frac{x^j q^{\left(j^2+j\right)/2}}{(q;q)_{t-j}(q;q)_{t+j}}$$

$$= \sum_{s,t\geq 0} \frac{q^{s^2+t^2}}{(q;q)_{s-t}(q;q)_{2t}} (-xq;q)_t \left(-x^{-1};q\right)_t$$

Rogers-Ramanujan identities are $x = -1$ and $x = -q$.

B-Z: A Short Rogers-Ramanujan Bijection, *Discrete Mathematics*, 1983

Generalized Rogers-Ramanujan Bijections, *Advances in Mathematics*, 1989

1983: Received preprint of Doron's proof of Andrews' conjecture (a.k.a. q-Dyson),

$$\text{C.T.} \prod_{1 \le i < j \le n} \left(\frac{x_i}{x_j}; q \right)_{a_i} \left(\frac{x_j}{x_i} q; q \right)_{a_j} = \frac{(q;q)_{a_1 + a_2 + \cdots + a_n}}{(q;q)_{a_1} (q;q)_{a_2} \cdots (q;q)_{a_n}}.$$

Basic idea goes back to Ira Gessel's proof of the Vandermonde determinant formula:

$$\prod_{1 \le i < j \le n} \left(x_i - x_j \right) = \sum_{\sigma \in S_n} (-1)^{\operatorname{sgn} \sigma} \prod_{i=1}^{n} x_i^{\sigma(i)}.$$

Interpret LHS as sum over tournaments and find a sign-reversing involution on non-transitive tournaments.

1982, Doron published proof of Dyson's conjecture based on Ira's idea, *Discrete Math*:

$$\text{C.T.} \quad \prod_{1\le i,j\le n}\left(1-\frac{x_i}{x_j}\right)^{a_i} = \binom{a_1+\cdots+a_n}{a_1,\ldots,a_n}.$$

LHS generates multi-tournaments, i and j play $a_i + a_j$ games, i wins a total of $(n - 1)\, a_i$ games, sign is determined by parity of number of upsets ($i < j$ but j beats i). RHS counts multi-words in a_1 1's, a_2 2's, …

$$\text{C.T.}\ \prod_{1\le i<j\le n}\left(\frac{x_i}{x_j};q\right)_{a_i}\left(\frac{x_j}{x_i}q;q\right)_{a_j}=\frac{(q;q)_{a_1+a_2+\cdots+a_n}}{(q;q)_{a_1}(q;q)_{a_2}\cdots(q;q)_{a_n}}.$$

Resulting words on RHS are weighted by Z-statistic. If W_{ij} is the subword in i and j, then

$$Z(W)=\sum_{i<j}\text{Maj}(W_{ij}),$$

Maj(W_{ij}) = sum of position of j's that are followed by i.

$$\text{C.T.} \prod_{1\le i<j\le n} \left(\frac{x_i}{x_j};q\right)_{a_i} \left(\frac{x_j}{x_i}q;q\right)_{a_j} = \frac{(q;q)_{a_1+a_2+\cdots+a_n}}{(q;q)_{a_1}(q;q)_{a_2}\cdots(q;q)_{a_n}}.$$

Resulting words on RHS are weighted by Z-statistic. If W_{ij} is the subword in i and j, then

$$Z(W) = \sum_{i<j} \text{Maj}(W_{ij}),$$

Maj(W_{ij}) = sum of position of j's that are followed by i.

The gap lay in the proof that the Z-statistic is Mahonian, that the sum over all words W in a_1 1's, a_2 2's, … of $q^{Z(W)}$ equals

$$\frac{(q;q)_{a_1+a_2+\cdots+a_n}}{(q;q)_{a_1}(q;q)_{a_2}\cdots(q;q)_{a_n}}$$

Z-B: A Proof of Andrews' q-Dyson Conjecture, *Discrete Math,* 1985

B. & Goulden: Constant Term Identities Extending the q-Dyson Theorem, *Transactions of **the** AMS*, 1985

———: The Generalized Plasma in One Dimension, *Communications in Mathematical Physics*, 1987

Gessel & Xin: A Proof of the Zeilberger-Bressoud q-Dyson Theorem, *Proceedings of the AMS*, 2006

1980's, Kathy O'Hara announced a somewhat complicated combinatorial proof that the Gaussian polynomial $(q;q)_{n+j}/(q;q)_n\,(q;q)_j$ is always unimodal. *J. Combinatorial Theory A*, 1990.

Doron showed that what Kathy actually proved is that

$$\begin{bmatrix} n+j \\ j \end{bmatrix} = \sum q^{m_1^2+\cdots+m_j^2-j} \prod_{i=1}^{j} \begin{bmatrix} (n+2)i - M_{i-1} - M_{i+1} \\ m_i - m_{i+1} \end{bmatrix},$$

$$m_1 \geq \cdots \geq m_j \geq m_{j+1} = 0, \quad m_0 = 0,$$

$$M_i = m_1 + \cdots + m_i, \quad M_j = j.$$

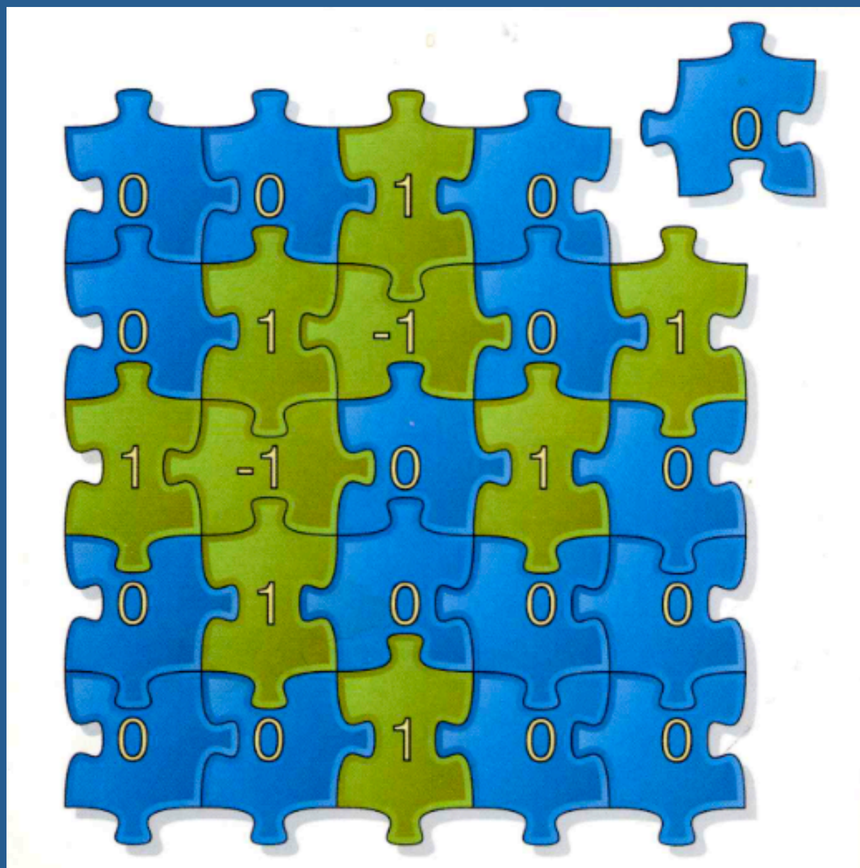
Unimodality follows by induction using the observation that each summand has the same mode.

B: In the Land of OZ, in *q-Series and Partitions*, IMA, 1989.

Interpretation and generalization of the O'Hara-Zeilberger identity.

Benjamin, Quinn, Quinn, and Wójs, Composite fermions and integer partitions, *J. Combin. Th. A*, 2001

Alternating Sign Matrices



Kuperberg's representation

David Robbins
(1942–2003)

					1					
				1		1				
			2		3		2			
		7		14		14		7		
	42		105		135		105		42	
429		1287		2002		2002		1287		429

$A_{n,k}$ = # of $n \times n$ alternating sign matrices with 1 in row 1, column k.

1

1 2/2 1

2 2/3 3 3/2 2

7 2/4 14 5/5 14 4/2 7

42 2/5 105 7/9 135 9/7 105 5/2 42

429 2/6 1287 9/14 2002 16/16 2002 14/9 1287 6/2 429

Conjecture: $$\frac{A_{n,k}}{A_{n,k+1}} = \frac{\binom{n-2}{k-1} + \binom{n-1}{k-1}}{\binom{n-2}{n-k-1} + \binom{n-1}{n-k-1}}$$

The conjecture about the ratios implies that

$$A_n = \prod_{j=0}^{n-1} \frac{(3j+1)!}{(n+j)!} = \frac{1!\cdot 4!\cdot 7!\cdots(3n-2)!}{n!\cdot(n+1)!\cdots(2n-1)!}$$

But this was George Andrews' formula for the number of descending plane partitions that fit into an $n\times n\times n$ box.

George's work was inspired by Ian Macdonald's conjectured generating function for cyclically symmetric plane partitions.

"If I had to single out the most interesting open problem in all of enumerative combinatorics, this would be it."
Richard Stanley, review of *Symmetric Functions and Hall Polynomials*, *Bulletin of the AMS*, March, 1981.

In the attempt to find a connection between descending plane partitions and alternating sign matrices that would prove their conjectures, Mills, Robbins, and Rumsey instead discovered a proof of Macdonald's conjecture.

MRR, Proof of the Macdonald Conjecture, *Inv. Math.*, 1982

Oberwolfach, 1982:

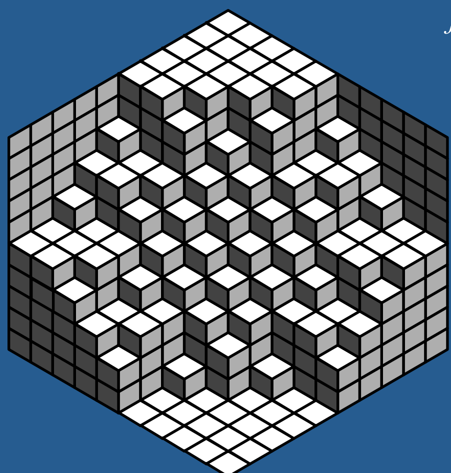
“Dave’s first talk, about the MRR proof of Macdonald’s CSPP conjecture, was so good, and it hinted at the intriguing ASMs, that Dominique [Foata], (and everyone else!) *begged* Dave to give a second fifty-minute talk, about ASMs and their conjectured enumeration…

“On the way back, I was fortunate to share a train cabin with Dave, and I asked him lots of questions, and thus started my love-hate relationship with the ASM conjecture.”

Doron, Dave Robbins’ art of guessing,
Advances in Applied Mathematics, 2005

1992: George Andrews proves Robbins conjecture that the number of totally symmetric, self-complementary plane partitions in an $n \times n \times n$ box is given by

$$A_n = \prod_{j=0}^{n-1} \frac{(3j+1)!}{(n+j)!} = \frac{1!\cdot 4!\cdot 7!\cdots(3n-2)!}{n!\cdot(n+1)!\cdots(2n-1)!}$$

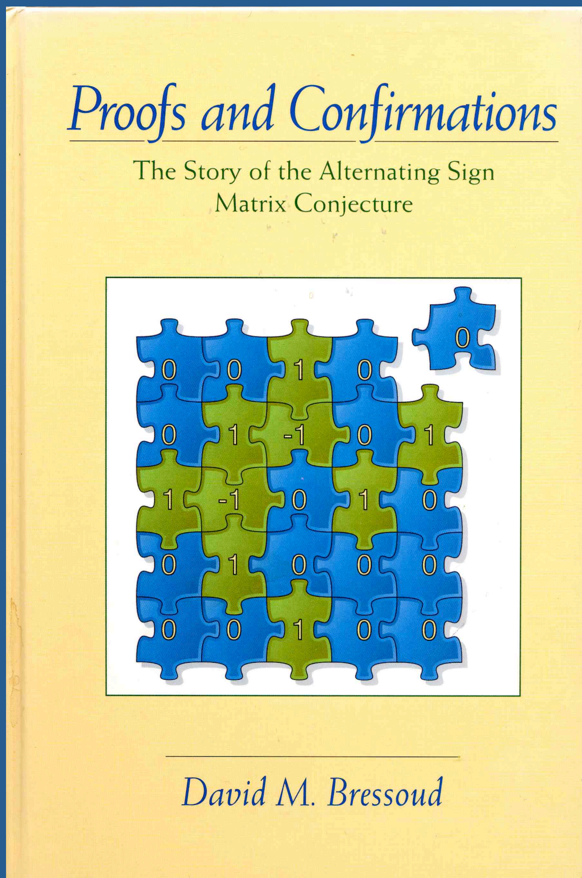


Z, Proof of the Alternating Sign Matrix Conjecture, *Elect. J. of Combin.*, 1996.

Greg Kuperberg, Another proof of the alternating sign matrix conjecture, *Int. Math. Res. Notices,* 1996.

Based on Rodney Baxter's triangle-to-triangle relationship, a basic tool of statistical mechanics.

Z., Proof of the refined alternating sign matrix conjecture, *New York J. of Math.*, 1996



Published jointly by MAA and Cambridge University Press, 1999

Concludes with Doron's brilliant proof of Robbins' original conjecture:

$$\frac{A_{n,k}}{A_{n,k+1}} = \frac{\binom{n-2}{k-1}+\binom{n-1}{k-1}}{\binom{n-2}{n-k-1}+\binom{n-1}{n-k-1}}$$

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