Proof of George Andrews' and David Robbins' q-TSPP Conjecture

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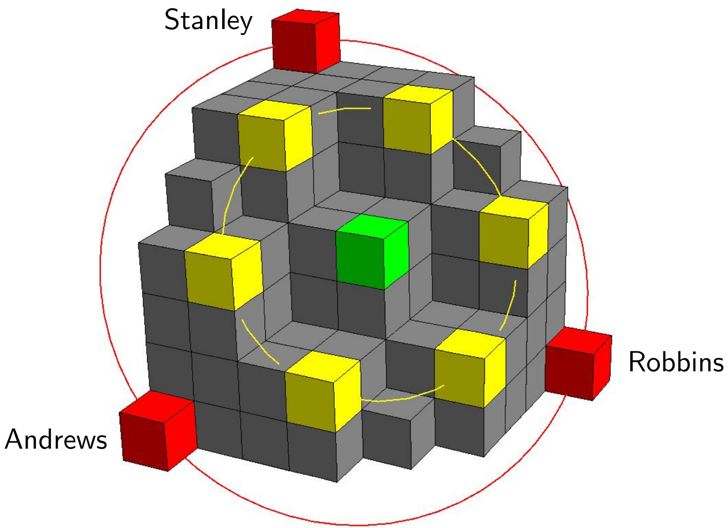
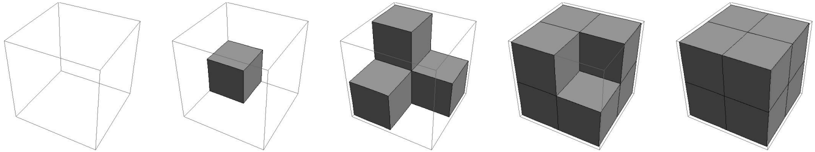
From $A = B$ to $Z = 60$
Conference in Honor of Doron Zeilberger's 60th Birthday
May 27, 2010

q-TSPP



Let $T(n)$ denote set of TSPPs with largest part at most n.



q^0 $\quad$ q^1 $\quad$ q^2 $\quad$ q^3 $\quad$ q^4

Andrews-Robbins q-TSPP conjecture:

$$\sum_{\pi \in T(n)} q^{|\pi / S_3|} = \prod_{1 \leq i \leq j \leq k \leq n} \frac{1 - q^{i+j+k-1}}{1 - q^{i+j+k-2}}$$

$\left(\equiv \right.$  $\left. \right)$

For $q = 1$:

$$|T(n)| = \prod_{1 \leq i \leq j \leq k \leq n} \frac{i + j + k - 1}{i + j + k - 2}$$

(Stembridge)

The determinant

Reduction by Soichi Okada:

The q-TSPP conjecture is true if  $\Big($  $\Big) \equiv$

$$\det(a_{i,j})_{1\le i,j\le n} = \prod_{1\le i\le j\le k\le n} \left(\frac{1-q^{i+j+k-1}}{1-q^{i+j+k-2}} \right)^2 =: b_n$$

where

$$a_{i,j} := q^{i+j-1} \left(\begin{bmatrix} i+j-2 \\ i-1 \end{bmatrix}_q + q \begin{bmatrix} i+j-1 \\ i \end{bmatrix}_q \right) + (1+q^i)\delta_{i,j} - \delta_{i,j+1}$$

where

$$\begin{bmatrix} n \\ k \end{bmatrix}_q := \frac{(1-q^n)(1-q^{n-1})\cdots(1-q^{n-k+1})}{(1-q^k)(1-q^{k-1})\cdots(1-q)}.$$

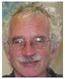
The holonomic ansatz

Second reduction by Doron Zeilberger:

“Pull out of the hat” a discrete function $c_{n,j}$ and prove

$$
\begin{aligned}
c_{n,n} &= 1 && (n \geq 1),\\
\sum_{j=1}^{n} c_{n,j} a_{i,j} &= 0 && (1 \leq i < n),\\
\sum_{j=1}^{n} c_{n,j} a_{n,j} &= \frac{b_n}{b_{n-1}} && (n \geq 1).
\end{aligned}
$$

Then $\det(a_{i,j})_{1\leq i,j\leq n} = b_n$ holds.

Hence,  () =  ( ()).

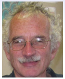
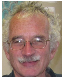
Pull out of the hat! ()

Manuel Kauers guessed some recurrences for $c_{n,j}$.

Their Gröbner basis has the form

$$\begin{aligned}
\bigcirc c_{n,j+4} &= \bigcirc c_{n,j} + \bigcirc c_{n,j+1} + \bigcirc c_{n,j+2} + \bigcirc c_{n,j+3} \\
&\quad + \bigcirc\, c_{n+2,j} + \bigcirc c_{n+2,j+1} \\
\bigcirc c_{n+1,j+3} &= \bigcirc c_{n,j} + \bigcirc c_{n,j+1} + \bigcirc c_{n,j+2} + \bigcirc c_{n,j+3} \\
&\quad + \bigcirc\, c_{n+1,j} + \bigcirc c_{n+1,j+1} + \bigcirc c_{n+1,j+2} \\
&\quad + \bigcirc\, c_{n+2,j} + \bigcirc c_{n+2,j+1} + \bigcirc c_{n+3,j} \\
\bigcirc c_{n+2,j+2} &= \bigcirc c_{n,j} + \bigcirc c_{n,j+1} + \bigcirc c_{n,j+2} + \bigcirc c_{n,j+3} \\
&\quad + \bigcirc\, c_{n+2,j} + \bigcirc c_{n+2,j+1} \\
\bigcirc c_{n+3,j+1} &= \bigcirc c_{n,j} + \bigcirc c_{n,j+1} + \bigcirc c_{n,j+2} + \bigcirc c_{n,j+3} \\
&\quad + \bigcirc\, c_{n+1,j} + \bigcirc c_{n+1,j+1} + \bigcirc c_{n+1,j+2} \\
&\quad + \bigcirc\, c_{n+2,j} + \bigcirc c_{n+2,j+1} + \bigcirc c_{n+3,j} \\
\bigcirc c_{n+4,j} &= \bigcirc c_{n,j} + \bigcirc c_{n,j+1} + \bigcirc c_{n,j+2} + \bigcirc c_{n,j+3} \\
&\quad + \bigcirc\, c_{n+2,j} + \bigcirc c_{n+2,j+1}
\end{aligned}$$

where each $\bigcirc$ represents a polynomial in $\mathbb{Q}[q, q^j, q^n]$ of total degree ≤ 100.

With  ( ( ())) it is now routine to prove  ( ()) (with the computer!):

$$\boxed{c_{n,n} = 1 \text{ for all } n \geq 1}$$

- recurrence for $c_{n,j}$ of the form

$$p_7 c_{n+7,j+7} = p_6 c_{n+6,j+6} + \cdots + p_1 c_{n+1,j+1} + p_0 c_{n,j}$$

  with $p_i \in \mathbb{Q}[q, q^j, q^n]$
- $j \to n$ yields a recurrence for the diagonal sequence $c_{n,n}$
- operator factors into $P_1 P_2$ with P_2 equivalent to $c_{n+1,n+1} = c_{n,n}$
- $c_{1,1} = \cdots = c_{7,7} = 1$

$$\boxed{(1+q^n) - c_{n,n-1} + \sum_{j=1}^{n} c'_{n,j} = \frac{b_n}{b_{n-1}}}$$

with $$c'_{n,j} = q^{n+j-1}\left(\begin{bmatrix} n+j-2 \\ n-1 \end{bmatrix}_q + q\begin{bmatrix} n+j-1 \\ n \end{bmatrix}_q\right) c_{n,j}$$

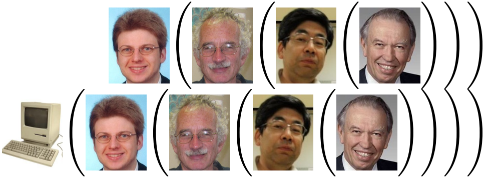
- recurrences for $c'_{n,j}$ via closure properties (HolonomicFunctions)
- find a relation of the form

$$p_7 c'_{n+7,j} + \cdots + p_1 c'_{n+1,j} + p_0 c'_{n,j} = t_{n,j+1} - t_{n,j}$$

where the $p_7, \ldots, p_0$ are rational functions in $\mathbb{Q}(q, q^n)$ and $t_{n,j}$ is a $\mathbb{Q}(q, q^j, q^n)$-linear combination of certain shifts of $c'_{n,j}$.

- creative telescoping: recurrence for the sum
- closure properties: recurrence for the left-hand side
- compare with recurrence for right-hand side
- (finitely many) initial values

Great! Are we done now?



= semi-rigorous proof

= proof

In practice, estimated timings are:

Zeilberger slow:	1677721600 days
Takayama:	52428800 days
Chyzak:	?
polynomial ansatz:	4000 days

New idea

Creative telescoping relation

$$p_7(q, q^n)c'_{n+7,j} + \cdots + p_0(q, q^n)c'_{n,j} = t_{n,j+1} - t_{n,j}$$

with $t_{n,j} = r_1(q, q^n, q^j)c'_{n+3,j+2} + \cdots + r_{10}(q, q^n, q^j)c'_{n,j}$

: ansatz with

$$r_k(q, q^n, q^j) = \frac{\sum_{l=0}^{L} r_{k,l}(q, q^n)(q^j)^l}{d_k(q, q^n, q^j)}$$

where the denominators d_k can be “guessed” (leading coefficients of the Gröbner basis).

$\longrightarrow$ linear system over $\mathbb{Q}(q, q^n)$

Modular computations

Techniques:

- polynomial interpolation
- rational reconstruction
- Chinese remaindering

Solving the linear system over $\mathbb{Q}(q, q^n)$:

- 1167 interpolation points for q
- 363 interpolation points for q^n
- each case takes about a minute (after lots of optimizations)
- estimated computation time: $1167 \cdot 363 \cdot 60s = 294$ days

Many other tricks...

Speedy Gonzales

The two compute servers at RISC

- speedy: QuadCore Xeon E5345, 9320MHz, 16GB RAM
- gonzales: QuadCore Xeon X5460, 12640MHz, 32GB RAM

worked day and night, interrupted only by the cleaning professionals and a RISC colleague. . .

The “birthday present”

 =

“Huge paper stack”

“Theorems for a price”?

Theorem for a prize!

Thanks, Doron, and happy birthday!