Ask Not What Doron Zeilberger Can Do For You; Ask What You Can Do For Doron

Aviezri S. Fraenkel
Weizmann Institute of Science
Rehovot, Israel

Beatty's Theorem, 1926; Rayleigh 1894

- For $\alpha>0$ irrational and $1/\alpha+1/\beta=1$, let
  $A=\cup_{n\geq 1} \{\lfloor n\alpha \rfloor\}$, $B=\cup_{n\geq 1} \{\lfloor n\beta \rfloor\}$.
- Then the sets A, B split the positive integers: $A\cap B=\emptyset$, $A\cup B=\mathbb{Z}_{\geq 1}$.
- Condition $1/\alpha+1/\beta=1$ is clearly necessary.
- The thm states that it's also sufficient.
- MAA, April 2010: "The result is so astonishing and yet easily proved that we include a short proof for the reader's pleasure."

Doron for Doron: a Pleasure Proof

- For any $k \in \mathbb{Z}_{\geq 1}$, number of terms < k is $\lfloor k/\alpha \rfloor + \lfloor k/\beta \rfloor$ (by irrationality of α)$= \lfloor k/\alpha \rfloor + \lfloor k(1-1/\alpha) \rfloor$ $= k + \lfloor k/\alpha \rfloor + \lfloor -k/\alpha \rfloor = k-1$.
- Similarly, $A \cup B$ contains k terms < k+1. Hence there is exactly one term < k+1 but not less than k; it equals k.

Application: P, N-positions in 2-player games

N-position: a position from which the Next player can force a win.

P-position: a position from which the Previous player can win.

$\mathcal{P}$, $\mathcal{N}$ – set of all P-positions, N-positions, respectively.

- P– Previous player can force a win.
- N– Next player can force a win. Thus:
- Position u $\in \mathcal{P}$ iff F(u) $\subseteq \mathcal{N}$.
- Position u $\in \mathcal{N}$ iff F(u) $\cap \mathcal{P} \neq \varnothing$.
- Notice that $\mathcal{P}$ and $\mathcal{N}$ are not symmetric.
- In the (directed) Game Graph, $\mathcal{P}$ is the graph kernel.
- The sets $\mathcal{P}$, $\mathcal{N}$ split $\mathbb{Z}_{\geq 1}$. Conversely, splittings into $\geq$2 sets often induce new games.

Wythoff's game

- Define a game on two piles of tokens:
- take any positive number of tokens from a single pile, or
- the same (positive) number of tokens from both piles.
- Player making last move wins.
- Then (0,0), **(1,2)**$\in\mathcal{P}$.

n	0	1	2	3	4	5	6	7	8	9	10
a_n	0	**1**	3	4	6	8	9	11	12	14	16
b_n	0	**2**	5	7	10	13	15	18	20	23	26

- Recursive winning strategy: $a_n = \text{mex}\,\{a_i, b_i : 0 \le i < n\}\ n \ge 0$, $b_n = a_n + n$.

Algebraic strategy:

- Let $\tau = (1+\sqrt{5})/2$, which is the solution of $1/x+1/(x+1)=1$; $\beta=\tau^2=\tau+1$.
- Theorem. $a_n=\lfloor n\tau \rfloor$, $b_n=\lfloor n\tau^2 \rfloor$ $n\geq 0$, and the squences $\{a_n\}$, $\{b_n\}$ are complementary for $n\geq 1$.
- Note: $\tau=[1,1,1,1,\ldots]$ (continued fraction expansion).
- Convergents: p_n/q_n, where
- $p_{-1}=p_0=1$, $p_n=p_{n-1}+p_{n-2}$ $(n\geq 1)$.

Exotic numeration system

- Every $N \in \mathbb{Z}_{\geq 1}$ has a unique representation: $N=\sum_{i\geq 0} d_i p_i$, where the digits d_i satisfy $0 \leq d_i \leq t$ ($i \geq 0$), and
- $d_i = t \Rightarrow d_{i-1}=0$ ($i \geq 1$). Then
- $R(a_n)$=all numbers ending in an even number of 0s, b_n all numbers ending in odd number of 0s; for every n, $R(b_n)$ is the <u>left shift</u> of $R(a_n)$.

Multi-pile Wythoff: illustration m=3

Take any positive number from a single pile, or a,b,c from the piles s.t.

(1) $a \oplus b \oplus c = 0$. ($\oplus$ is Nim-sum; this is generalization of Wythoff.)

- Write the P-positions in the form $C_j = (j, A_n^j, B_n^j)_{n \geq 0}$, $1 \leq j \leq A_n^j \leq B_n^j$, j fixed. Claim: Under the move rule (1), Wythoff strategy is almost preserved:
- A_n^j, B_n^j almost split $\mathbb{Z}_{\geq 1}$
- A_n^j is almost mex $\{A_i^j, B_i^j : i<n\}$
- $B_n^j - A_n^j = 1$ $\forall$ large n.

Explaining "almost preservation"

- For j=1, $(1,2,k)\in\mathcal{N}\ \forall\, k\geq 2$, since $(1,2)\in\mathcal{P}$ in Wythoff. Thus 2 cannot appear in the list of P-positions of 3-pile Wythoff.
- A small set X of integers is excluded.
- How does this affect, if at all, the structure of the complementary sequences?

Two conjectures (F, 1993)

(1) For every fixed $j \geq 1$, $\exists$ integer n_j and finite set $X=X^j \subset \mathbb{Z}_{\geq 0}$, s.t. $\forall\ n \geq n_j$,

- $A_n^j = \text{mex}\,(X^j \cup \{A_i^j, B_i^j : 0 \leq i < n\})$, $B_n^j = A_n^j + n$.
- (2) For every fixed $j \geq 1$, $\exists$ integer γ_j, s.t. $\forall\ n \geq n_j$,
- $A_n^j \in \{\lfloor n\tau \rfloor - \gamma_j - 1,\ \lfloor n\tau \rfloor - \gamma_j,\ \lfloor n\tau \rfloor - \gamma_j + 1\}$,

- For approaching the conjectures, investigated a splitting system perturbed by X:

Recall: $a_n = \text{mex}\{a_i, b_i : 0 \leq i < n\}$ $n \geq 0$, $b_n = a_n + n$, $A = \{a_n\}_{n \geq 1}$, $B = \{b_n\}_{n \geq 1}$. Then $A = \{\lfloor n\tau \rfloor\}_{n \geq 1}$, $B = \{\lfloor n\tau^2 \rfloor\}_{n \geq 1}$.

Let $X \subsetneq \mathbb{Z}_{\geq 1}$, X finite,

$a'_n = \text{mex}_1 \{X \cup \{a'_i, b'_i : n_0 \leq i < n\}\}$, $b'_n = a'_n + n$, $n \geq n_0$,

$A' = \{a'_n\}_{n \geq n0}$, $B' = \{b'_n\}_{n \geq n0}$.

Let N=max (X)+1. Then A′, B′ are N-upper complementary for some $n_0 \geq 1$.

Relate A′ to A, B′ to B.

Shift sequence: $s_n := a_n - a'_n$

- Theorem (Krieger, F 2004). $\exists\ p \in \mathbb{Z}_{\geq 1}$, $\gamma \in \mathbb{Z}$ s.t. $\forall\ n \geq p$, either $s_n = \gamma$; or else $\forall\ n \geq p$, $s_n \in \{\gamma-1, \gamma, \gamma+1\}$. If the latter then
- s_n assumes each of the 3 values infinitely often,
- $s_n \neq \gamma \Rightarrow s_{n-1} = s_{n+1} = \gamma$.
- Indices of irregular shifts can be partitioned into K subsets, each of which satisfies a linear recurrence.

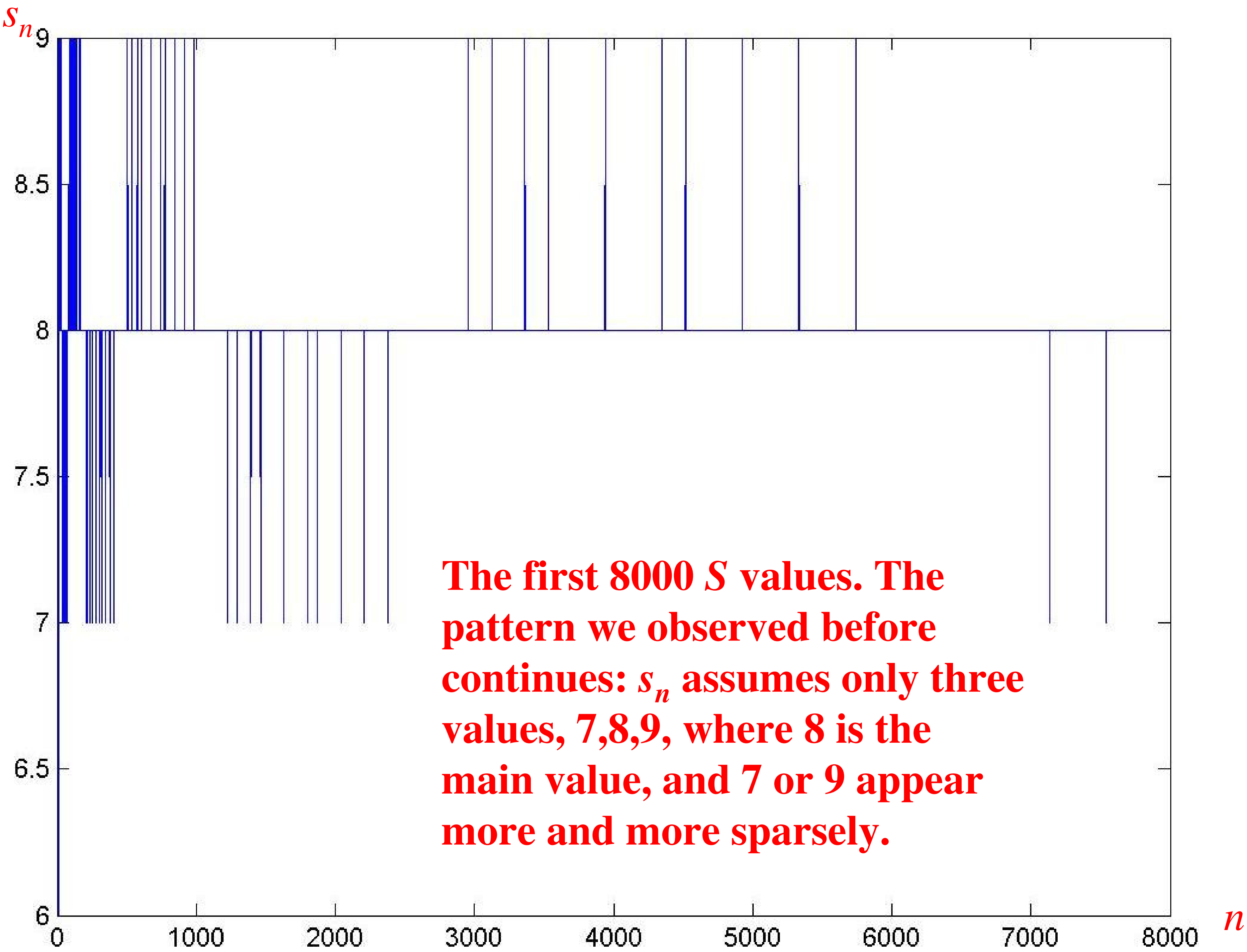
More on the conjectures

- We also proved Conjecture (1) $\Rightarrow$ Conjecture (2).
- Was also proved by Xinyu Sun 2007 with additional results.
- Zeilberger, Sun (2004) proved the 2 conjectures for $m=3$ and $1 \leq j \leq 10$.

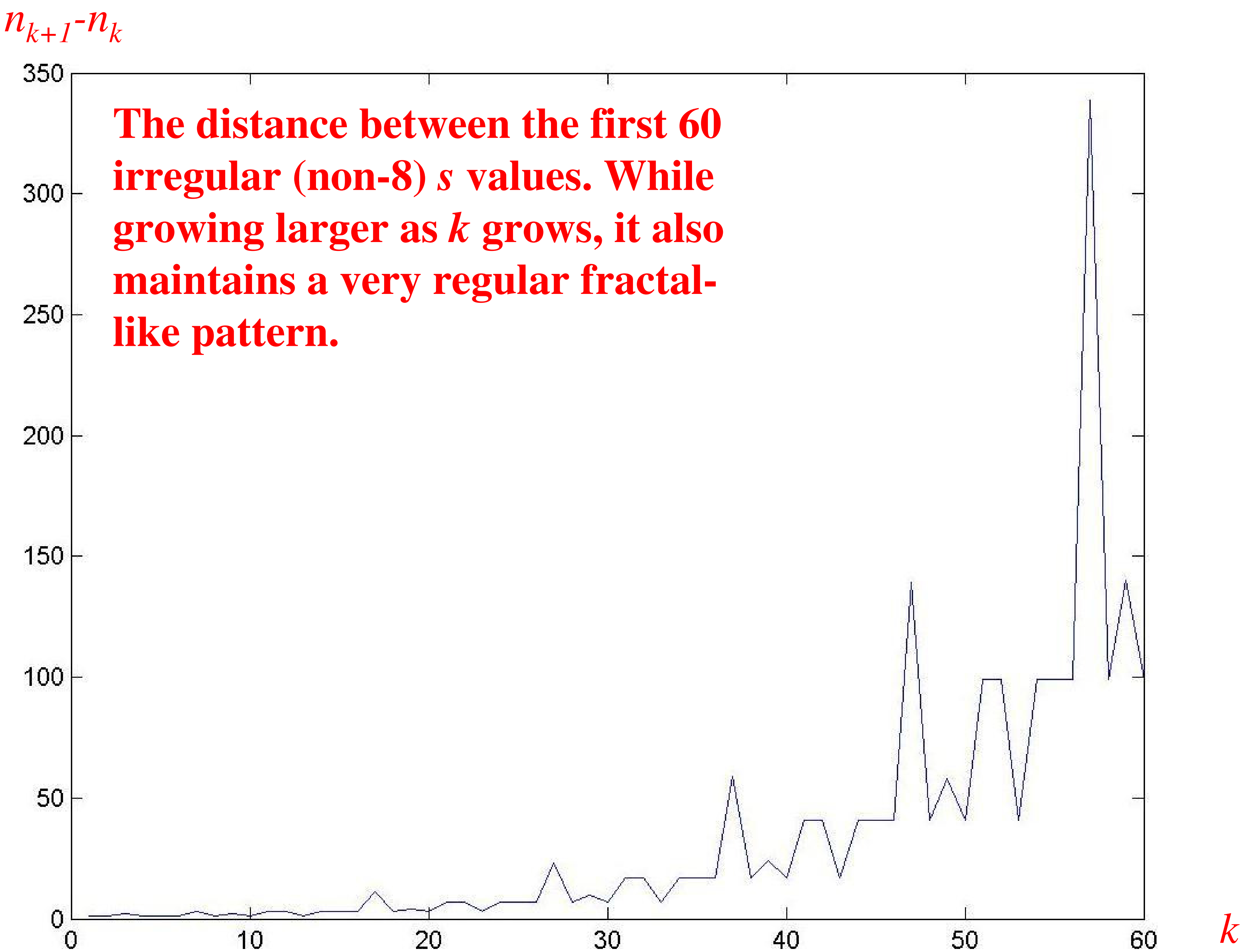
Example: t = 2, n_0 = 6, X = {1,5}.

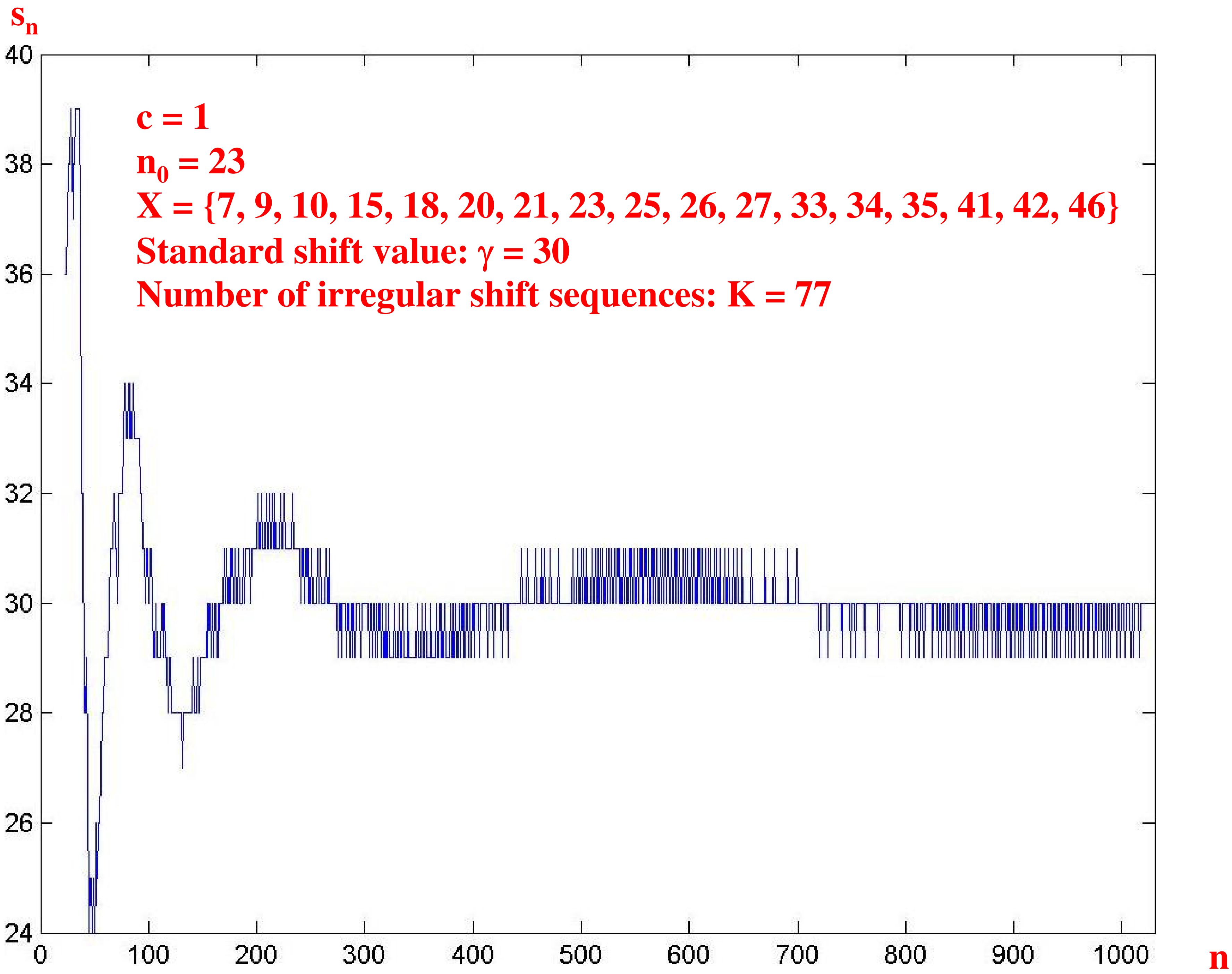
n	a_n	b_n	s_n	a'_n	b'_n
6	2	14	6	8	20
7	3	17	6	9	23
8	4	20	7	11	27
9	6	24	6	12	30
10	7	27	7	14	34
11	8	30	7	15	37
12	9	33	7	16	40
13	10	36	8	18	44
14	11	39	8	19	47
15	12	42	9	21	51
16	13	45	9	22	54
17	15	49	9	24	58
18	16	52	9	25	61
19	18	56	8	26	64
20	19	59	9	28	68

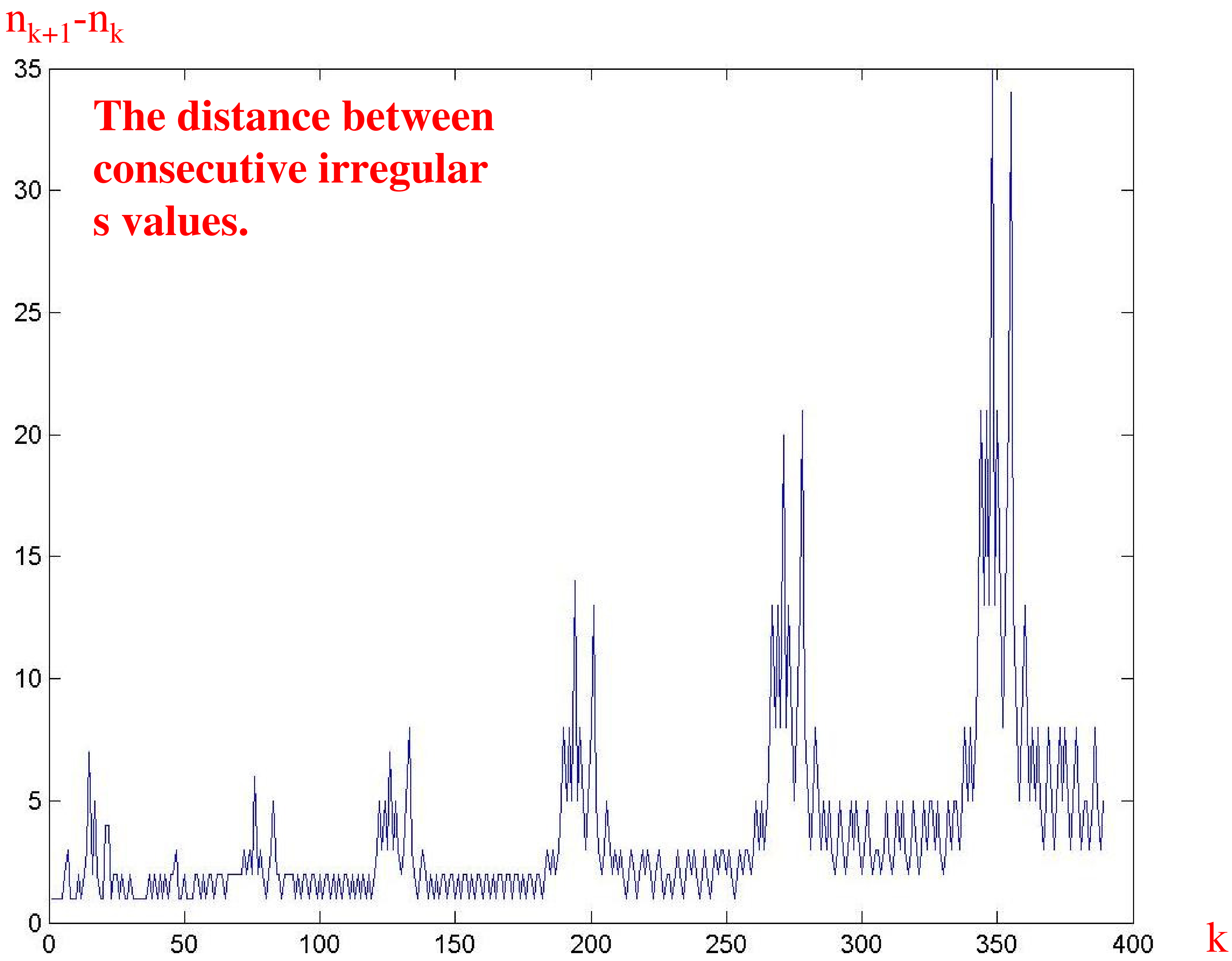
n	a_n	b_n	s_n	a'_n	b'_n
21	21	63	8	29	71
22	22	66	9	31	75
23	23	69	9	32	78
24	25	73	8	33	81
25	26	76	9	35	85
26	28	80	8	36	88
27	29	83	9	38	92
28	31	87	8	39	95
29	32	90	9	41	99
30	34	94	8	42	102
31	35	97	8	43	105
32	37	101	8	45	109
33	38	104	8	46	112
34	40	108	8	48	116
35	41	111	8	49	119



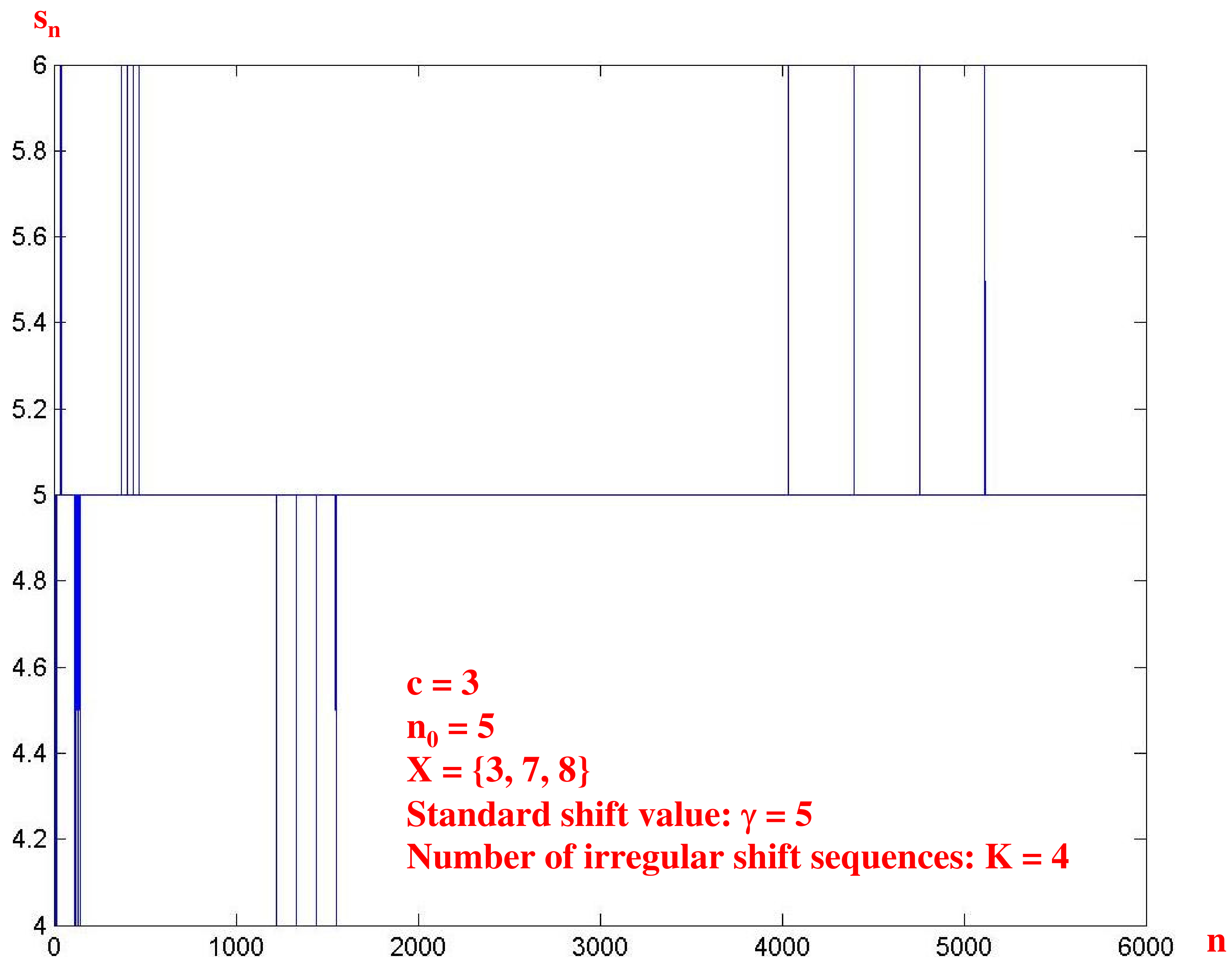
The first 8000 *S* values. The pattern we observed before continues: s_n assumes only three values, 7,8,9, where 8 is the main value, and 7 or 9 appear more and more sparsely.


$n_{k+1}-n_k$
The distance between the first 60 irregular (non-8) *s* values. While growing larger as *k* grows, it also maintains a very regular fractal-like pattern.
350
300
250
200
150
100
50
0
0
10
20
30
40
50
60
k

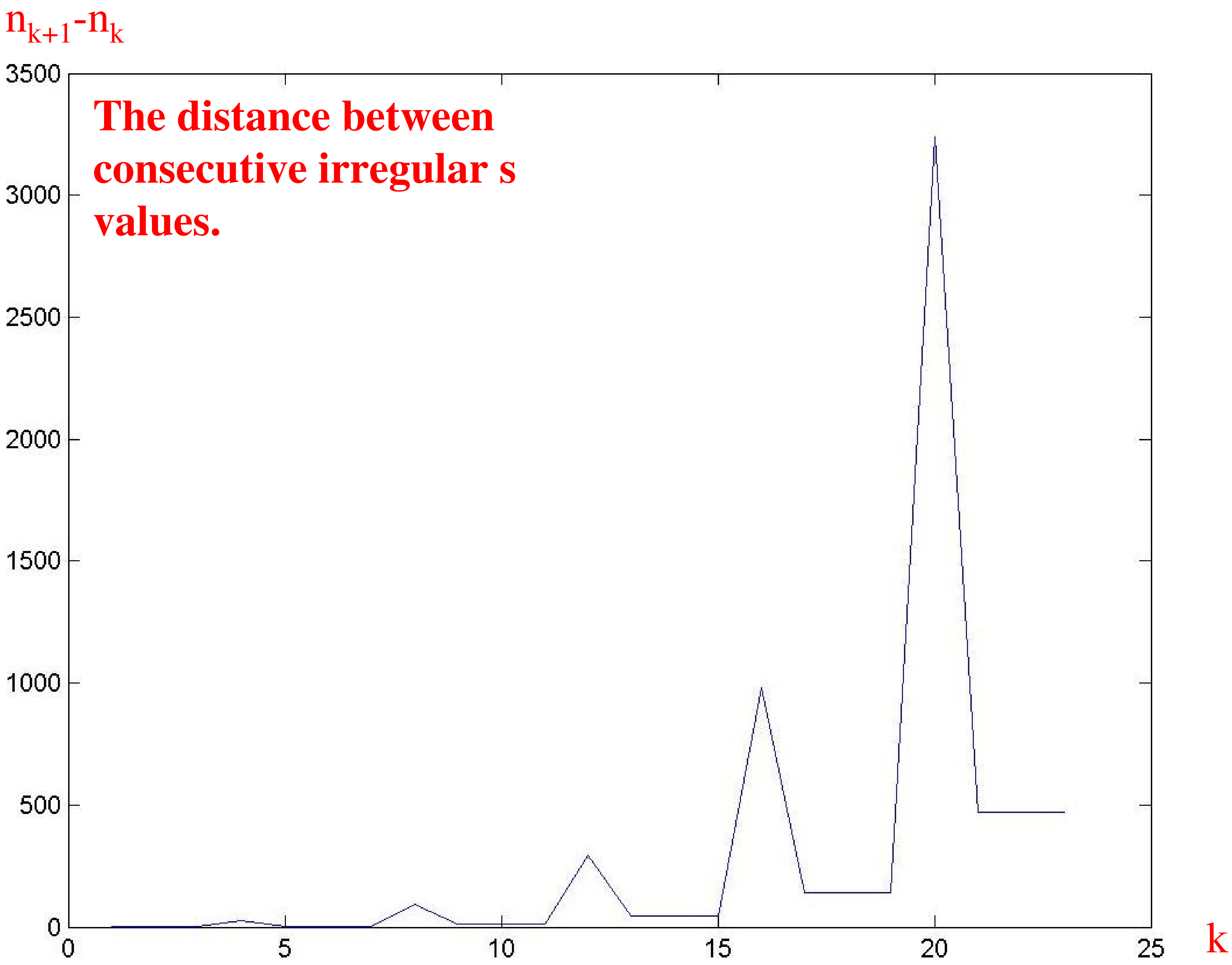

s_n
c = 1
$n_0 = 23$
X = {7, 9, 10, 15, 18, 20, 21, 23, 25, 26, 27, 33, 34, 35, 41, 42, 46}
Standard shift value: γ = 30
Number of irregular shift sequences: K = 77
n
40
38
36
34
32
30
28
26
24
0
100
200
300
400
500
600
700
800
900
1000



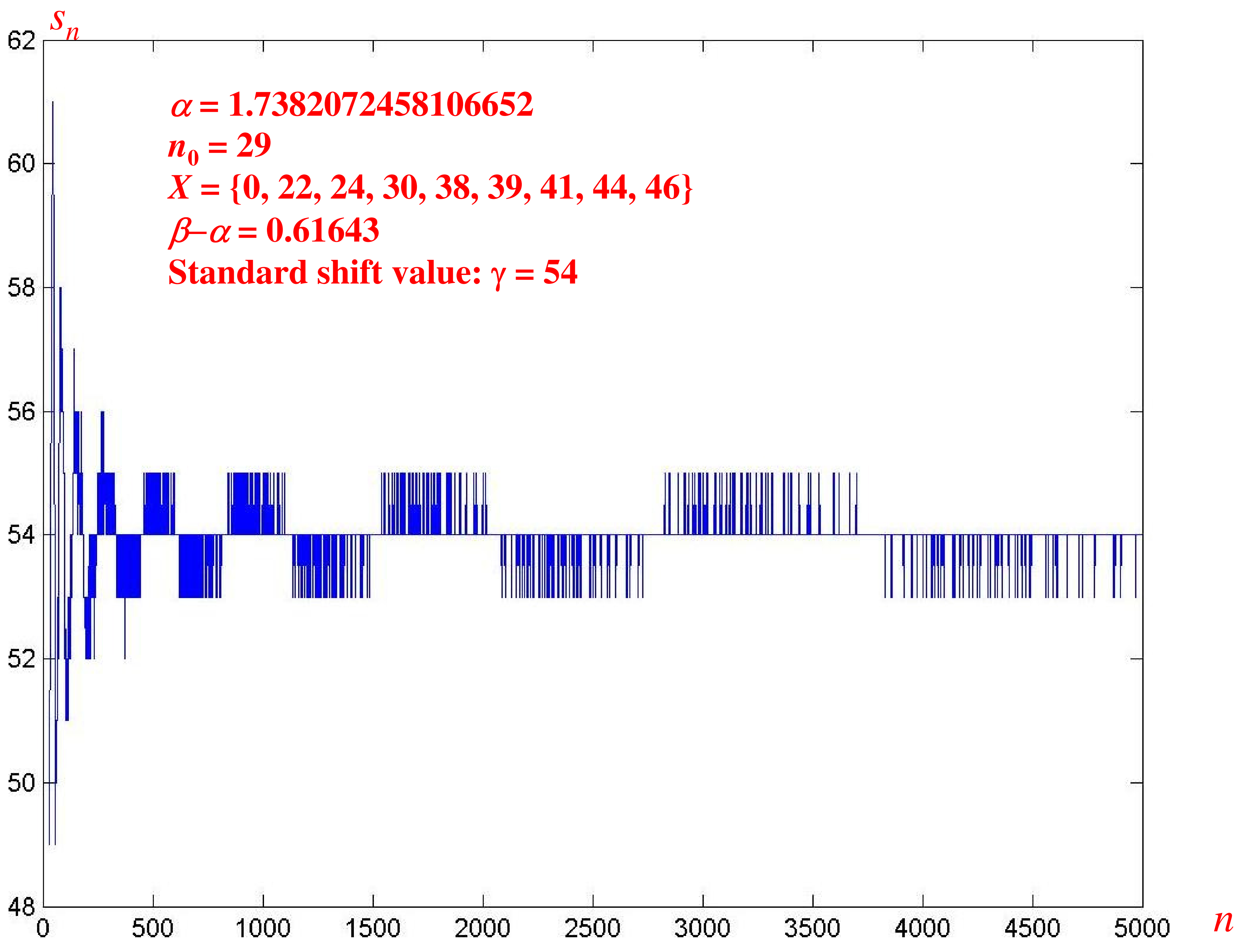
n_{k+1}-n_k
The distance between
consecutive irregular
s values.
35
30
25
20
15
10
5
0
0
50
100
150
200
250
300
350
400
k



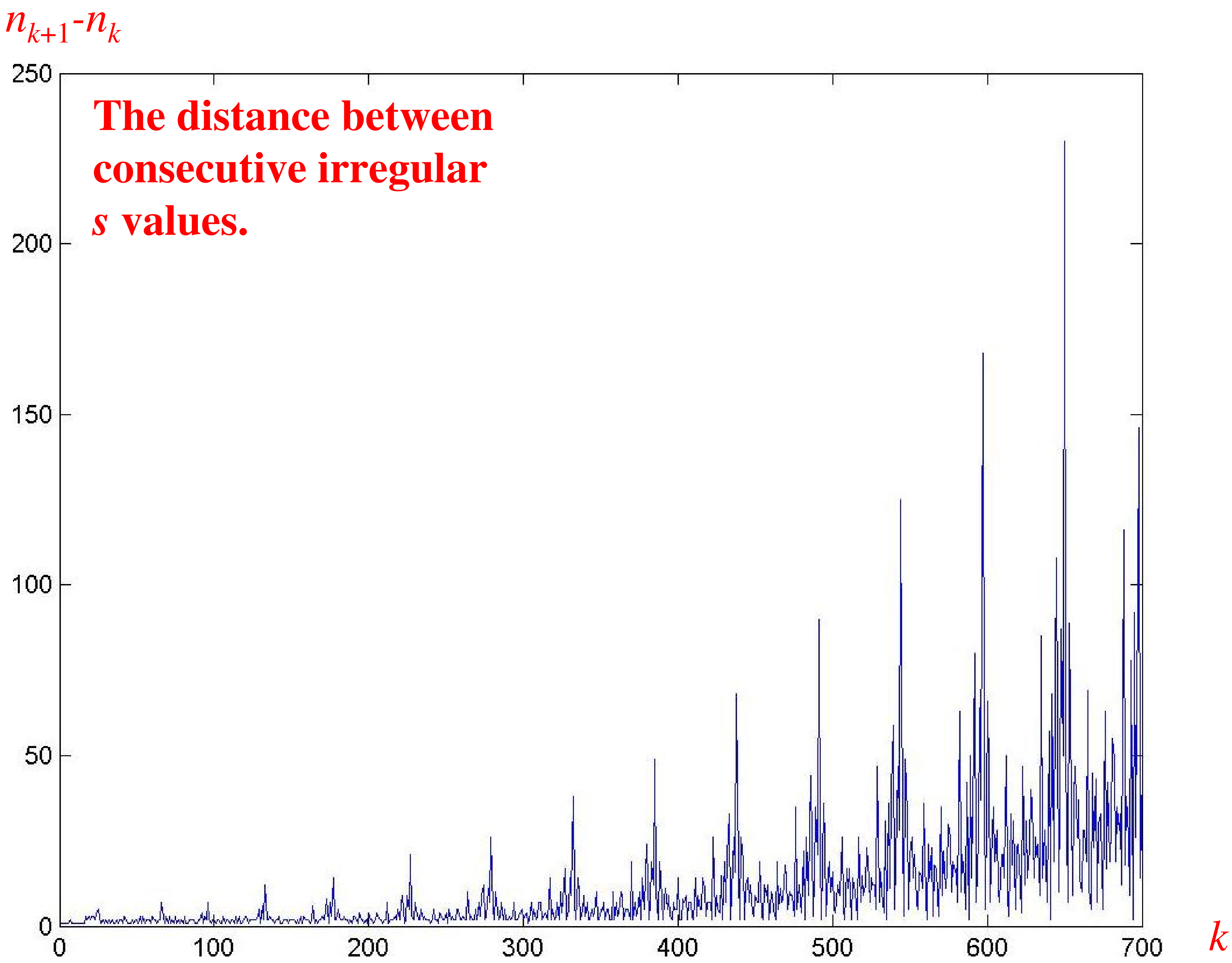
s_n
6
5.8
5.6
5.4
5.2
5
4.8
4.6
4.4
4.2
4
0
1000
2000
3000
4000
5000
6000
n
c = 3
$n_0 = 5$
X = {3, 7, 8}
Standard shift value: $\gamma = 5$
Number of irregular shift sequences: K = 4



$n_{k+1}-n_k$
The distance between consecutive irregular s values.
3500
3000
2500
2000
1500
1000
500
0
0
5
10
15
20
25
k



s_n
$\alpha = 1.7382072458106652$
$n_0 = 29$
$X = \{0, 22, 24, 30, 38, 39, 41, 44, 46\}$
$\beta - \alpha = 0.61643$
Standard shift value: $\gamma = 54$
62
60
58
56
54
52
50
48
0
500
1000
1500
2000
2500
3000
3500
4000
4500
5000
n



n_{k+1}-n_k
The distance between consecutive irregular *s* values.
250
200
150
100
50
0
0
100
200
300
400
500
600
700
k

Questions

- What determines the number K of irregular shift sequences s_n?
- For some perturbation sets X get $\gamma_n=n$ for all n, without getting the additional two values $\gamma_n=n-1$, $\gamma_n=n+1$. Characterize those cases.
- Perturbation sets for general Beatty sequences.

Continued

- Same questions for s-fold complementarity and fractional complementarity..

Doron and Xinyu Sun 2004

- Proved the two conjectures for 3 piles and $1 \leq j \leq 10$.
- Thus if you do something for Doron, you get at least a 10-fold return. Moreover, there is the prospect of an ∞-return.
- They wrote: The method discussed here should be extendable to prove the conjectures for Wythoff's games with more than 3 heaps. A numerical method, instead of the symbolic one presented here, may also be developed to improve the performance…

Doron and Xinyu (contnd)

- …We hope the result presented here would be a stepping-stone for others to finally prove the conjectures, and better yet, to provide a *constructive* polynomial-time winning strategy for the game.

S-fold complementarity

- Let $s \in \mathbb{Z}_{\geq 1}$. Cover every positive integer exactly s times.
- Theorem (O'Bryant 2002, Larsson 2010). α irrational and $1/\alpha+1/\beta=s$, $\alpha<\beta$. Let

  $$A=\cup_{n\geq s}\{\lfloor n\alpha \rfloor\},\ B=\cup_{n\geq 1}\{\lfloor n\beta \rfloor\}.$$

- Then the sets A, B s-split the positive integers: $A\cup B=s\times\mathbb{Z}_{\geq 1}$.

Proof of s-fold Beatty Theorem

- O'Bryant: Generating function, Power series.
- Larsson: Combinatorial.
- Pleasure proof of AMM can be extended easily to s-fold complementarity.

Uspensky 1927, Graham 1963

- $\alpha_1,\ldots,\alpha_m$ positive real numbers. Suppose that $\lfloor n\alpha_1 \rfloor_{n\geq 1},\ldots,\lfloor n\alpha_m \rfloor_{n\geq 1}$ split the positive integers. Then $m \leq 2$.
- Uspensky's proof depends on Kronecker's Theorem on simultaneous diophantine approximation. Graham's is purely combinatorial.

Another Question

- Does Uspensky and Graham's result hold also for s-fold complementarity?
- We (Hegarty, Larsson, F) conjecture that the answer is positive, excepting trivial cases.

A conjecture solved for the integers, irrationals; wide open for the rationals.

- Let $0<\alpha_1<\alpha_2<\ldots<\alpha_m$, $\gamma_1,\ldots,\gamma_m$ reals, $m\geq 3$. If $\cup_{i=1}^m \lfloor n\alpha_i+\gamma_i\rfloor$ $(n\geq 1)$ is a DCS, then $\alpha_i=(2^m-1)/2^{m-i}$, $i=1,\ldots,m$ (F 1973).
- Easy to see that

$\cup_{i=1}^m \lfloor n(2^m-1)/2^{m-i}\rfloor-2^{i-1}+1$, $i=1,\ldots,m$, $n\geq 1$, is indeed a DCS. Example: $m=3$.

$\bigcup_{i=1}^{m} \lfloor n(2^m-1)/2^{m-i} \rfloor - 2^{i-1}+1,\ i=1,\ldots,m,\ n \geq 1$

n	$\lfloor 7n/4 \rfloor$	$\lfloor 7n/2 \rfloor - 1$	$7n-3$
1	1	2	4
2	3	6	
3	5		
4	7		

Split with arithmetic sequences

- Evens and odds; evens and numbers =1 mod 4, numbers=3 mod 4…´ Theorem. Suppose that $\cup_{i=1}^{m} (na_i+b_i)$, $n \geq 1$, $m \geq 2$, is a DCS, $0<a_1 \leq \ldots \leq a_m$ integers. Then $a_{m-1}=a_m$.
- Proof. Consider the generating function $\sum_{i=1}^{m} z^{bi}/(1-z^{ai})=z/(1-z)$.

Mirsky, Newman, Davenport, Rado

- Proof. $\sum_{i=1}^{m} z^{b_i}/(1-z^{a_i})=z/(1-z)$. Suppose $a_{m-1}<a_m$. Let ξ=primitive a_mth root of unity, and let $z\to\xi$. (in Erdos 1952). Erdos asked for elementary proof.
- 1^{st} elementary proof: Berger, Felzenbaum, F 1986. Others followed.

Irrational case

- $\alpha>0$ irrational, $1/\alpha+1/\beta=1$. Then $\{\lfloor n\alpha\rfloor\}_{n\geq1}$ and $\{\lfloor n\beta\rfloor\}_{n\geq1}$ split $\mathbb{Z}_{\geq1}$. So do $\{\lfloor n\alpha\rfloor\}_{n\geq1}$, $\{\lfloor(2n)\beta)\rfloor\}_{n\geq1}$, $\{\lfloor(2n-1)\,\beta\rfloor\}_{n\geq1}$.
- Graham 1973: All irrational DCS are compositions of integer DCS.
- So 2 moduli are same for $m\geq3$. Only the rational case is left open.

- Erdos & Graham, 1980. Special cases: F, 1973; Simpson, 1991; Berger, Felzenbaum, F, 1986, Morikawa, 1982, 1985 proved for m=3. Morikawa, 1985, Simpson 2004: "Japanese Remainder Theorem". Simpson, 1991: conjecture true if $\alpha_1 \leq 3/2$. Altman, Gaujal, Hordijk 2000: proved for m=4, using "balanced sequences". Using same method, Tijdeman proved m=5, 1998; m=6, 2000; Barat, Varju m=7 (2005).

- Using balanced sequences, Tijdeman proved for m=5, 1998; m=6, 2000; Barat, Varju for m=7, 2005. Graham O’Bryant, 2005 generalized conjecture to s-covering, used Fourier analysis to prove special cases. Vuillon, 2003; Paquin, Vuillon, 2007.
- Scheduling: Kubiak, 2003; Brauner, Crama, 2004; Brauner, Jost, 2008.