Pattern Avoidance and Alternating Sign Matrices

Arvind Ayyer

Institut de Physique Théorique

CEA Saclay, France

From A=B to Z=60, Rutgers University, 28 May, 2010

DZ has the distinction of being well-known among two sets of species

DZ has the distinction of being well-known among two sets of species

1. Humans - Fairly famous

DZ has the distinction of being well-known among two sets of species

1. Humans - Fairly famous

2. Computers - Legendary

DZ has the distinction of being well-known among two sets of species

1. Humans - Fairly famous

2. Computers - Legendary

Unfortunately, we have not been able to program computers enough to be able to celebrate a human's birthday. That's an assignment for us with your 80th birthday as a deadline!

Wish you Many Many Happy Returns !

Outline

1. Monotone Triangles
2. 312-avoiding permutations
3. Relation between 1) and 2)
4. Gog Words
5. Subpatterns for gog words
6. Relation between 1) and 5)
7. “Sub-bijection” between gogs and magogs

Monotone Triangles

A *monotone triangle/ strict Gelfand pattern/ gog triangle* of size n is an array $\pi = (a_{ij})$ of positive integers defined for $n \geq i \geq j \geq 1$ that is written in the form

$$\begin{array}{llll} a_{1,1} & & & \\ a_{2,1} & a_{2,2} & & \\ \vdots & \vdots & \ddots & \\ a_{n,1} & a_{n,2} & \cdots & a_{n,n}, \end{array}$$

where,

1. $a_{i,j} < a_{i,j+1}$, $a_{i,j} \leq a_{i-1,j}$ and $a_{i,j} \leq a_{i+1,j+1}$.

2. $a_{n,j} = j$.

The numbers increase along rows, decrease weakly along columns, and decrease weakly along diagonals. The last row is the integers from 1 to n.

Example: $n = 4$

$$\begin{array}{llll} 4 & & & \\ 3 & 4 & & \\ 1 & 3 & 4 & \\ 1 & 2 & 3 & 4, \end{array}$$

corresponds naturally to the permutation 4312.

There is a natural bijection between monotone triangles and alternating sign matrices (MRR, 1983); thus counted by

$$A_n = \prod_{j=0}^{n-1} \frac{(3j+1)!}{(n+j)!}.$$

A monotone triangle a is said to contain a *gap at row* i if there exists a j such that

$$a_{i,j} - a_{i+1,j} > 1.$$

In the above example, there is a gap at row 2 because $a_{2,1} = 3$.

312 Patterns

A permutation π of the letters $\{1, \dots, n\}$ *contains the pattern 312* if there exists positive integers i, j, k, with $1 \leq i < j < k \leq n$ such that $\pi_i > \pi_k > \pi_j$. If no such triple of integers exist, the permutation π is said to *avoid the pattern 312*.

Macmahon (1915) showed that 123-avoiding permutations of size n are counted by the Catalan number C_n. So also are all other permutations avoiding any other pattern of size three.

Remark about 312 patterns

If a permutation π contains the pattern 312 at positions (i, j, k) and $j - i > 1$, then there exists another integer l, $i \leq l \leq j$ such that the permutation contains another pattern 312 at $(l, l+1, k)$.

The proof of this remark comes by looking at the integer π_{i+1}. If $\pi_{i+1} < \pi_k$ then $(i, i+1, k)$ is the required triple. If not, then $(i+1, j, k)$ is the new triple and we iterate.

For example, the permutation $\pi = \underline{5}42\underline{1}6\underline{3}$ has a 312 pattern as shown. The iteration is as follows,

$$\underline{5}42\underline{1}6\underline{3} \to 5\underline{4}2\underline{1}6\underline{3} \to 5\underline{4}\underline{2}16\underline{3}.$$

312-containing permutations and monotone triangles

Lemma 1. *A permutation π on n letters contains a 312 pattern if and only if there exists a gap in the corresponding monotone triangle a. Moreover, a 312 pattern is at positions $(i, i+1, j)$ if and only if there is a gap at row i.*

In the previous example, 4312, the 312 pattern is at $(2,3,4)$ and the gap is between $a_{2,1}$ and $a_{3,1}$. Look at $3\underline{7}56\underline{4}12$

$$
\begin{array}{lllllll}
3 & & & & & & \\
3 & \underline{7} & & & & & \\
3 & 5 & 7 & & & & \\
3 & 5 & 6 & 7 & & & \\
\underline{3} & 4 & 5 & 6 & 7 & & \\
1 & 3 & 4 & 5 & 6 & 7 & \\
1 & 2 & 3 & 4 & 5 & 6 & 7,
\end{array}
$$

with two gapped rows as shown.

Central Idea of proof

A permutation π does not have the 312 pattern at $(l, l+1, k)$ if and only if

$$\{\pi_j | j \leq l, \pi_j \geq \pi_l\} = \{\pi_j, \pi_j + 1, \ldots, \pi_j + p\}$$

The gog alphabet

The *gog alphabet*, $\mathcal{A}$, is given by tuples $(\alpha_1, \ldots, \alpha_{2p-1})$ such that each $1 \le \alpha_1 < \cdots < \alpha_{2p-1}$ and p is a positive integer.

For example, $\mathcal{A}$ contains all the integers but in addition contains elements such as $(1, 2, 3)$ and $(7, 9, 10, 13, 17)$.

In 2003, just before the Iraq War, former President George W. Bush tried once more to get the support of France. His approach to French President Jacques Chirac was straightforward, drawing on thousands of years of history and on higher authority: "This confrontation is willed by God, who wants to use this conflict to erase his people's enemies before a New Age begins." This, he added, was the "holy" war in the Middle East, predicted in the Bible, against Gog and Magog.

Gog Words

A *gog word* w of size n is a word consisting of elements of $\mathcal{A}$ satisfying the following conditions:
(a) The length w is n,
(b) Each tuple in w has length at most n and max. at most n,
(c) An integer in an even numbered position in a tuple is repeated in another tuple to its left and to its right in odd numbered positions,

Clearly, gog words with tuples of size one correspond to permutations.

For example 2(123)2 and 213 are both gog words of size 3 and 25(12456)(345)(234)3 is a gog word of size 6.

Bijection with ASMs

25(12456)(345)(234)3 corresponds to the matrix,

$$\begin{bmatrix} 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 1 & -1 & 0 & 1 & -1 & 1 \\ 0 & 0 & 1 & -1 & 1 & 0 \\ 0 & 1 & -1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \end{bmatrix}$$

and to the monotone triangle

$$\begin{array}{llllll} 2 \\ 2 & 5 \\ 1 & 4 & 6 \\ 1 & 3 & 5 & 6 \\ 1 & 2 & 4 & 5 & 6 \\ 1 & 2 & 3 & 4 & 5 & 6 \end{array}$$

Subsequences for gog words

An integer n is *active* with respect to a tuple $x = (\alpha_1, \ldots, \alpha_{2p-1})$ if $n > \alpha_{2p-1}$ or if $\alpha_{2k-1} < n < \alpha_{2k}$ for $k \in [p-1]$.

For example, 2 and 5 are active with respect to (134) but 3 is not active with respect to (124).

The integers a, b, c form a *subsequence* of the gog word $w = x_1 x_2 \ldots x_n$ if
(a) a, b, c appear in odd positions in x_i, x_j, x_k where $i < j < k$,
(b) c is not in an even position in $x_{i+1}, \ldots, x_{k-1}$,
(c) c is active with respect to x_j.

312-subpatterns for gog words

A gog word $w = x_1 \dots x_n$ contains the *312-subpattern* if it contains a subsequence (c, a, b) with $a < b < c$. If no such subsequence exists, we say that the gog word *avoids the subpattern 312*.

$\underline{31}(\underline{2}34)3$ contains the 312-subpattern.

$2\underline{5}(\underline{1}2356)5\underline{4}2$ ✓

contains a 312-subpattern but

$2\underline{5}(\underline{1}2456)5\underline{3}2$ ✗

does not.

Remarks about 312 subpatterns

OBVIOUS: If $w = x_1 \ldots x_n$ contains the 312-subpattern (c, a, b)

$$\ldots \underbrace{(\ldots, c, \ldots)}_{x_i} \ldots \underbrace{(\ldots, a, \ldots)}_{x_j} \ldots \underbrace{(\ldots, b, \ldots)}_{x_k} \ldots,$$

then it also contains a 312-subpattern (c', a', b)

$$\ldots \underbrace{(\ldots, \ldots, c')}_{x_i} \ldots \underbrace{(a', \ldots, \ldots)}_{x_j} \ldots \underbrace{(\ldots, b, \ldots)}_{x_k} \ldots.$$

NOT SO OBVIOUS: If a gog word $w = x_1 \ldots x_n$ contains the 312-subpattern, (c, a, b) appears in x_i, x_{i+1}, x_k.

$$\ldots \underbrace{(\ldots, c, \ldots)}_{x_i} \ldots \underbrace{(\ldots, a, \ldots)}_{x_j} \ldots \underbrace{(\ldots, b, \ldots)}_{x_k} \ldots,$$

then it also contains a 312-subpattern (c', a', b)

$$\ldots \underbrace{(\ldots, \ldots, c')}_{x_i'} \underbrace{(a', \ldots, \ldots)}_{x_{i'+1}} \ldots \underbrace{(\ldots, b, \ldots)}_{x_k} \ldots.$$

Main result: 312-subpatterns and monotone triangles

A gog word w of size n contains the generalized 312 pattern if and only if there exists a gap in the corresponding monotone triangle a. Moreover, a 312 pattern is present in tuples $(i, i+1, j)$ if and only if there is a gap at row i.

David Robbins quote:

"These conjectures are of such compelling simplicity that it is hard to understand how any mathematician can bear the pain of living without understanding why they are true."

These could have well referred to the relation between ASMs, DPPs and TSSCPPs.

Magogs A *magog triangle or fundamental domain for partitions* of size n is an array (b_{ij}) of positive integers defined for $n \geq i \geq j \geq 1$ that is written in the form

$$\begin{array}{llll} b_{1,1} & & & \\ b_{2,1} & b_{2,2} & & \\ \vdots & \vdots & \ddots & \\ b_{n,1} & b_{n,2} & \cdots & b_{n,n}, \end{array}$$

where,

1. $b_{i,j} \leq n$,

2. $b_{i,j} \leq b_{i+1,j}$ and $b_{i,j} \leq b_{i,j+1}$ whenever both sides are defined (weak decrease along both directions),

3. $b_{i,j} \geq i$.

Example: $n = 4$

$$\begin{array}{llll} 1 & & & \\ 1 & 3 & & \\ 3 & 3 & 4 & \\ 4 & 4 & 4 & 4. \end{array}$$

There is a natural bijection between magog triangles and totally symmetric self-complementary plane partitions (MRR, 1986).

A magog triangle is said to contain a *gap at row* i if there exists a j such that

$$b_{i+1,j} - b_{i,j} > 1.$$

In the above example, there is a gap between $b_{2,1}$ and $b_{3,1}$.

We do not know a natural structure on TSSCPPs which gives rise to gaps in magog triangles.

For example, there are no gaps in magog triangles of size 2 and for size 3, there is only one gapped magog triangle which corresponds to the TSSCPP

$$\begin{bmatrix} 6 & 6 & 6 & 4 & 3 & 3 \\ 6 & 6 & 6 & 4 & 3 & 3 \\ 6 & 6 & 4 & 3 & 2 & 2 \\ 4 & 4 & 3 & 2 & 0 & 0 \\ 3 & 3 & 2 & 0 & 0 & 0 \\ 3 & 3 & 2 & 0 & 0 & 0 \end{bmatrix}$$

“Subbijection” between gapless gogs and magogs

Given a gapless gog triangle $a = (a_{i,j})$,

$$b_{i,j} = a_{i,j} + i - j$$

is a gapless magog triangle

Given a gapless magog triangle $b = (b_{i,j})$,

$$a_{i,j} = b_{i,j} - i + j$$

is a gapless gog triangle.

The proof is just definition-checking!

Example: $n = 6$

$$\begin{array}{llllll} 3 \\ 2 & 4 \\ 1 & 3 & 4 \\ 1 & 3 & 4 & 5 \\ 1 & 2 & 3 & 4 & 5 \\ 1 & 2 & 3 & 4 & 5 & 6 \end{array} \quad \longleftrightarrow \quad \begin{array}{llllll} 3 \\ 3 & 4 \\ 3 & 4 & 4 \\ 4 & 5 & 5 & 5 \\ 5 & 5 & 5 & 5 & 5 \\ 6 & 6 & 6 & 6 & 6 & 6 \end{array}$$

Open questions

1. Other pattern avoiding structures?

2. Structure on the TSSCPP side?

3. Generalization of the bijection?

4. How many are in bijection? Enumerative formula? Asymptotics?

and finally...

Teaser

Count the number of bijected ASMs according to the position of the 1 in the bottom row and the total number

$$
\begin{array}{ccccccccccc}
 & & & & & 1 & & & & & \\
 & & & & 1 & & 1 & & & & \\
 & & & 2 & & 2 & & 2 & & & \\
 & & 6 & & 8 & & 6 & & 6 & & \\
 & 26 & & 44 & & 40 & & 26 & & 26 & \\
162 & & 332 & & 360 & & 272 & & 162 & & 162
\end{array}
\quad \longleftrightarrow \quad
\begin{pmatrix} 1 \\ 2 \\ 6 \\ 26 \\ 162 \\ 1450 \end{pmatrix}
$$

Count the number of bijected ASMs according to the position of the 1 in the top row

$$
\begin{array}{ccccccccccc}
 & & & & & 1 & & & & & \\
 & & & & 1 & & 1 & & & & \\
 & & & 2 & & 3 & & 1 & & & \\
 & & 6 & & 12 & & 7 & & 1 & & \\
 & 26 & & 66 & & 54 & & 15 & & 1 & \\
162 & & 504 & & 532 & & 220 & & 31 & & 1
\end{array}
$$